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OPTIMIZATION OF SYSTEMS WITH SEVERAL COST FUNCTIONALS

by

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A THESIS

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ABSTRACT

The optimization of systems with several cost functionals, (both single and multi-control cases) have been treated in this thesis. While the discussion has been mostly restricted to deterministic systems, some results for systems with sudden changes in parameters are included.

It is shown that the optimization of a system with respect to an objective function (which is expressed as a function of several given cost functionals) is embedded in the linear combination problem (i.e.) an optimization problem in which the performance criterion is a linear combination of the given cost functionals. According to the technique proposed in this thesis, the optimal controls are first determined for the linear combination problem as a function of the weighting factors. A search technique is then used to determine the optimum values of the weighting factors for the given objective function. One of the advantages of this technique is that the solution of the linear combination problem can be used for different objective functions formed out of the same set of cost functionals. Examples are included to show the application of this technique.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER (1)

INTRODUCTION

1.1 Background

For the past two or three decades much attention has been devoted to the problem of optimizing the behavior of systems. This has resulted in a variety of mathematical formulations and computational techniques. Some examples of such problems are maximizing the range of a rocket, minimizing the fuel consumed to achieve some required terminal state, maximizing the profit of a business, and minimizing the error in the estimation of the position of a certain object. Searching for a control which satisfies all the requirements and at the same time minimizes (or maximizes) a performance criterion constitutes the fundamental problem of optimization theory. The salient feature of this approach is that the performance criterion described by a single functional and a single control function is used. However, there are many real-world situations which cannot be meaningfully described by the above approach. A description involving a single cost functional with several controls or several cost functionals with a single control or several cost functionals with several controls may be necessary to describe the situation. Ho[7]* has classified some of these situations.

In recent years several investigators have attempted to tackle problems of this type. For an optimization problem with a single control and several costs some of the references are[6,11,12,15,16].

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* Numbers in [ ] refer to references listed at the end of the Chapter. Note that references appear at the end of each Chapter.

Optimization problems of the differential game type (zero-sum games) in which a single cost functional and two control functions appear are discussed in[1,2,8,9,13] among others. However, not much work appears to have been done on the optimization of systems governed by several controls and several cost functionals (n-players-games or non-zero sum games), especially from an engineering point of view. The references which the author has come across in this subject are[3,4,5,10,14].

1.2 Scope of the Thesis

In this thesis it is proposed to discuss the following types of problems.

a) Optimization of a deterministic system with a single control function and several cost functionals (see figure (1.1)). This problem is discussed in Chapter (2).

b) Optimization of a deterministic system with two control functions (zero-sum differential game) and several cost functionals is discussed in Chapter (3), (see figure (1.2)).

c) In Chapter (4), the optimization of a linear system subject to a linear quadratic performance criterion is discussed. Two antagonistic control functions will be used. It will be assumed that the parameters as well as the weighting matrices in the performance index are subject to sudden changes in values. This problem will be tackled as stochastic version of a special case of the problem discussed in Chapter (3).

d) The optimization of a deterministic system with several

Given

System Model: $\dot{X} = f(X,U)$

Cost Scales: $J_1,\dots,J_N$

Search for $U^* \varepsilon \, \mathcal{U}$ such that

$$\phi(J_1,\dots,J_N)\Big|_{U^*\varepsilon\,\mathcal{U}} \leq \phi(J_1,\dots,J_N)\Big|_{U\varepsilon\,\mathcal{U}}$$

$\forall U \varepsilon \, \mathcal{U}$

$\mathcal{U}$ is the set of admissible controls

Objective

Figure (1.1) Optimization of a System with a Single Control Function with Respect to an Objective Function (minimization problem)

Given

System Model: $\dot{X} = f(X,U,V)$

Cost Scales: $J_1,\ldots,J_N$

Search for $U^* \varepsilon \mathcal{U}$ and $V^* \varepsilon \mathcal{V}$, such that

$$\phi(J_1,\ldots,J_N)\Big|_{(U,V^*)} \leq \phi(J_1,\ldots,J_N)\Big|_{(U^*,V^*)} \leq \phi(J_1,\ldots,J_N)\Big|_{(U^*,V)}$$

$\forall U \varepsilon \mathcal{U}$ and $V \varepsilon \mathcal{V}$

$\mathcal{U}$ & $\mathcal{V}$ are sets of admissible controls

Objective

Figure (1.2) Optimization of a System with Two Control Functions, with Respect to an Objective Function (Min-Max Problem)

control functions and several cost functionals (non-cooperative games) is discussed in Chapter (5), (see figure (1.3)).

The format of each chapter is essentially the same and consists of the following:

a) Summary of previous work and scope of the rest of the chapter,

b) Mathematical formulation. Here necessary conditions for optimality are obtained using Pontryagin's Maximum Principle,

c) Discussion of computational techniques used,

d) Application of the technique to a few examples, and,

e) Discussion of results.

Only the titles and the broad outline of the problems considered in this thesis are mentioned here. A more detailed description of each of the problems and its relationship to previous work in the area will be found at the beginning of each chapter.

Given

System Model: $\dot{X} = f(X, U_1, \ldots, U_p)$

Cost Scales: $J_1, \ldots, J_{N_k}$,

$k = 1, \ldots, p$

Search for $U_k^* \varepsilon \mathcal{U}_k$, such that

$$\phi_k(J_1, \ldots, J_{N_k})\Big|_{(U_1^*, .., U_k^*, .., U_p^*)} \leq \phi_k(J_1, \ldots, J_{N_k})\Big|_{(..U_{k-1}^*, U_k, U_{k+1}^* ..)}$$

$$\forall U_k \varepsilon \mathcal{U}_k$$

$$k = 1, \ldots, p$$

Objective

Figure (1.3) Optimization of a System with p-Control Functions and p-Objective Functions.

REFERENCES

1. Behn, R.D. and Ho, Y.C. "On a Class of Linear Stochastic Differential Games", IEEE Trans. Automatic Control, Vol. AC-13, pp. 227-240, June 1968.

2. Berkowitz, L.D., "A Variational Approach to Differential Games", in Annals of Mathematical Studies, No. 52. Princeton, N.J.: Princeton, University Press 1964.

3. Case, J.H., "Towards A Theory of Many Player Differential Games", MRC, U.S. Army, The University of Wisconsin, Technical Summary Report #839, Jan 1968.

4. Case, J.H., "On Some Differential Games in Economics", MRC, U.S. Army, The University of Wisconsin, Technical Summary Report #874, May 1968.

5. Case, J.H., "Towards a Theory of Many Player Differential Games", SIAM J.on Control,Vol. 7, No. 2, pp. 179-197, May 1969.

6. Chang, S.S. L., "General Theory of Optimal Processes", SIAM J. on Control, Vol. 4, No. 1, pp. 46-55, February 1966.

7. Ho, Y.C., "Towards Generalized Control Theory", IEEE Trans. Automatic Control, Vol. AC-14, pp. 753-754, December 1969.

8. Ho, Y.C., Bryson, A.E., Jr., and Baron, S., "Differential Games and Optimal Pursuit - Evasion Strategies", IEEE Trans. Automatic Control, Vol. AC-10, pp. 385-389, October 1965.

9. Isaacs, R., Differential Games , New York: John Wiley 1965.

10. Karvovskiy, G.S. and Kuznetsov, A.D., "The Maximum Principle in Differential Games for N Players", Engineering Cybernetics, No. 6, pp. 10-13, Nov.-Dec. 1966.

11. Nelson, W.L., "On the Use of Optimization Theory for Practical Control System Design", IEEE Trans. Automatic Control, Vol. AC-9, pp. 469-478, October 1964.

12. Rekasius, Z.V. and Hsia, T.C., "On the Inverse Problem in Optimal Control", IEEE Trans. Automatic Control, Vol. AC-9, pp. 370-374, October 1964.

13. Rhodes, I.B. and Luenberger, D.G., "Differential Games with Imperfect State Information", IEEE Trans. Automatic Control, Vol. AC-14, pp. 29-38, February 1969.

14. Sarma, I.G., Ragade, R.K., Prasad, U.R., "Necessary Conditions for Optimal Strategies in a Class of Noncooperative N-Person Differential Games", J. SIAM Control, Vol. 7, No. 4, pp. 637-644, November 1969.

15. Waltz, F.M., "An Engineering Approach: Hierarchical Optimization Criteria", IEEE Trans. Automatic Control, Vol. AC-12, pp. 179-180, April 1967.

16. Zadeh, L.A., "Optimality and Non-Scalar-Valued Performance Criteria", IEEE Trans. Automatic Control, Vol. AC-8, pp. 59-60, January 1963.

CHAPTER (2)

OPTIMIZATION OF A DETERMINISTIC SYSTEM WITH A SINGLE CONTROL FUNCTION AND SEVERAL COST FUNCTIONALS

2.1 Background

One of the fundamental requirements for any optimization technique to be effective, is that the optimality criterion must some how reflect in an analytic form and in proper proportions all of the factors which are apparently important to a given system. Several approaches have been suggested for the problem of optimization of a dynamical system with a single control function and several cost functionals. Some of them are summarized below.

Nelson[7] has proposed the following technique (see figure 2.1). Suppose we have several cost scales $J_1,\ldots,J_N$. Let us designate the most important cost by J_N. Then the problem is to search for the control function $U^*(t)$ which minimizes J_N subject to the constraints ($J_k \leq \beta_k$, $k = 1,\ldots N$, β_k's are constants) at final time. The key to this approach lies in the choice of J_N and the constants β_k's, which may not always be an easy task. However, on the whole, this technique is considered a useful approach.

Waltz[11] has proposed what he calls "Hierarchical Optimization Criteria" and his technique can be explained as follows, (see figure 2.2). The given costs are ranked in order of importance as J_1, $J_2,\ldots,J_N$. We first optimize with respect to the primary cost J_1 and get the optimum value of this cost say V_1. The second step is to replace V_1 by $V_1 + \Delta V_1$ where $\Delta V_1 > 0$, and optimize the system with

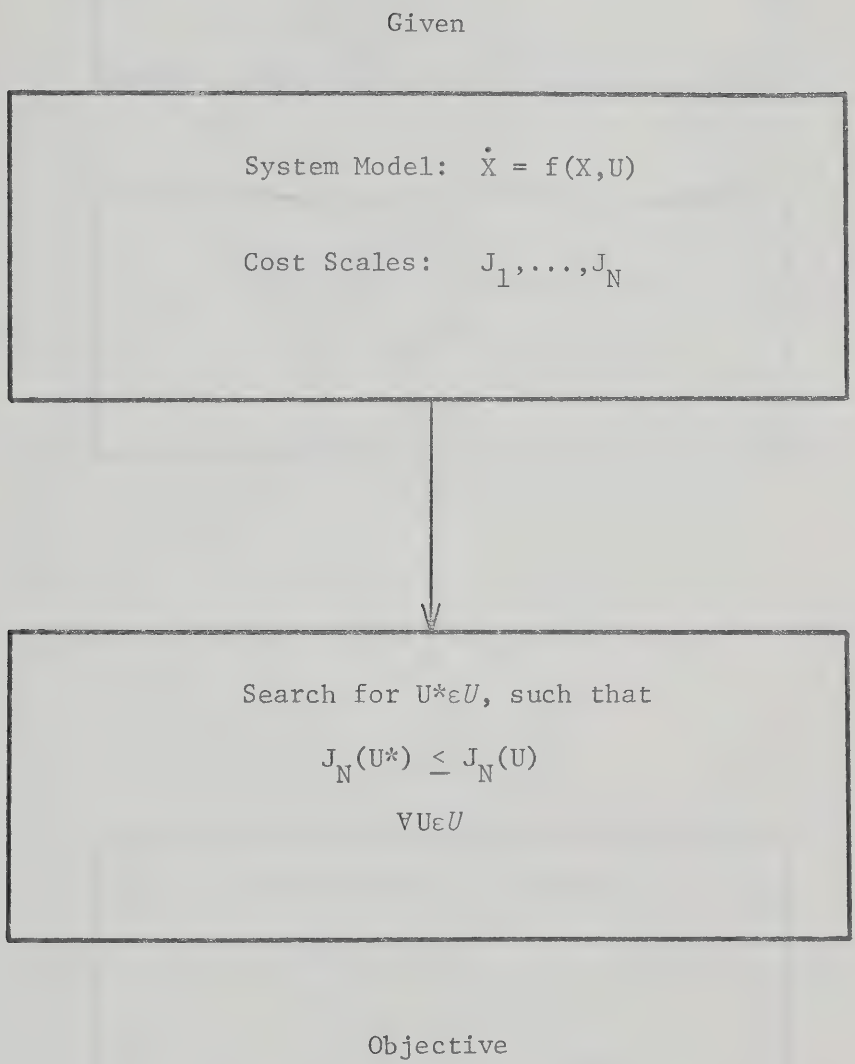


Figure (2.1) Nelson's Technique

Given

System Model: $\dot{X} = f(X,U)$

Objective

Search for $U^* \varepsilon \mathcal{U}$, such that

$$V_1 = J_1(U^*) \leq J_1(U)$$

$$\forall U \varepsilon \mathcal{U}$$

Given

System Model $\dot{X} = f(X,U)$

$$J_1(U) \leq V_1 + \Delta V_1$$

$$\vdots$$

$$J_{N-1}(U) \leq V_{N-1} + \Delta V_{N-1}$$

Objective

Search for $U^* \varepsilon \mathcal{U}$, such that

$$V_N = J_N(U^*) \leq J_N(U)$$

$$\forall U \varepsilon \mathcal{U}$$

Figure (2.2) Waltz's Technique

respect to the second cost J_2, at the same time ensuring that $J_1(U) \leq V_1 + \Delta V_1$, and get the optimum value of the second cost say V_2. The third step is to change V_2 by an increment ΔV_2, and optimize with respect to the third cost with the restriction that $J_1(U) \leq V_1 + \Delta V_1$ and $J_2(U) \leq V_2 + \Delta V_2$ and so on. This technique is essentially a constructive approach to choose the β_k's in the previous technique but the disadvantage of the present method is the increments $\Delta V_1, \ldots, \Delta V_N$, must be chosen by the designer and this is a matter of engineering judgement, moreover we have N sequential optimization problems.

A novel approach is due to Zadeh[12] in which he has introduced the term "Vector-Valued Cost". Chang has used Zadeh's idea and obtained a generalized version of Pontryagin's Maximum Principle[3]. See figure (2.3). In this approach Chang ends up with a problem involving the minimization of a linear combination of the given costs with all the weighting factors being positive. Unlike the previous methods, all the cost scales are considered simultaneously instead of one at a time. This is an improvement. However, to the best knowledge of this author, Chang has not described any technique to determine the weighting factors c_k's. These factors must be known in order to implement the optimal control strategy in any practical application.

The work reported in the remaining portion of this chapter although closely related to Chang's work goes further. The optimal control is determined for a performance criterion which in turn is expressed as a function of several cost scales. This criterion will be referred to as the "Objective Function". The basis for choosing

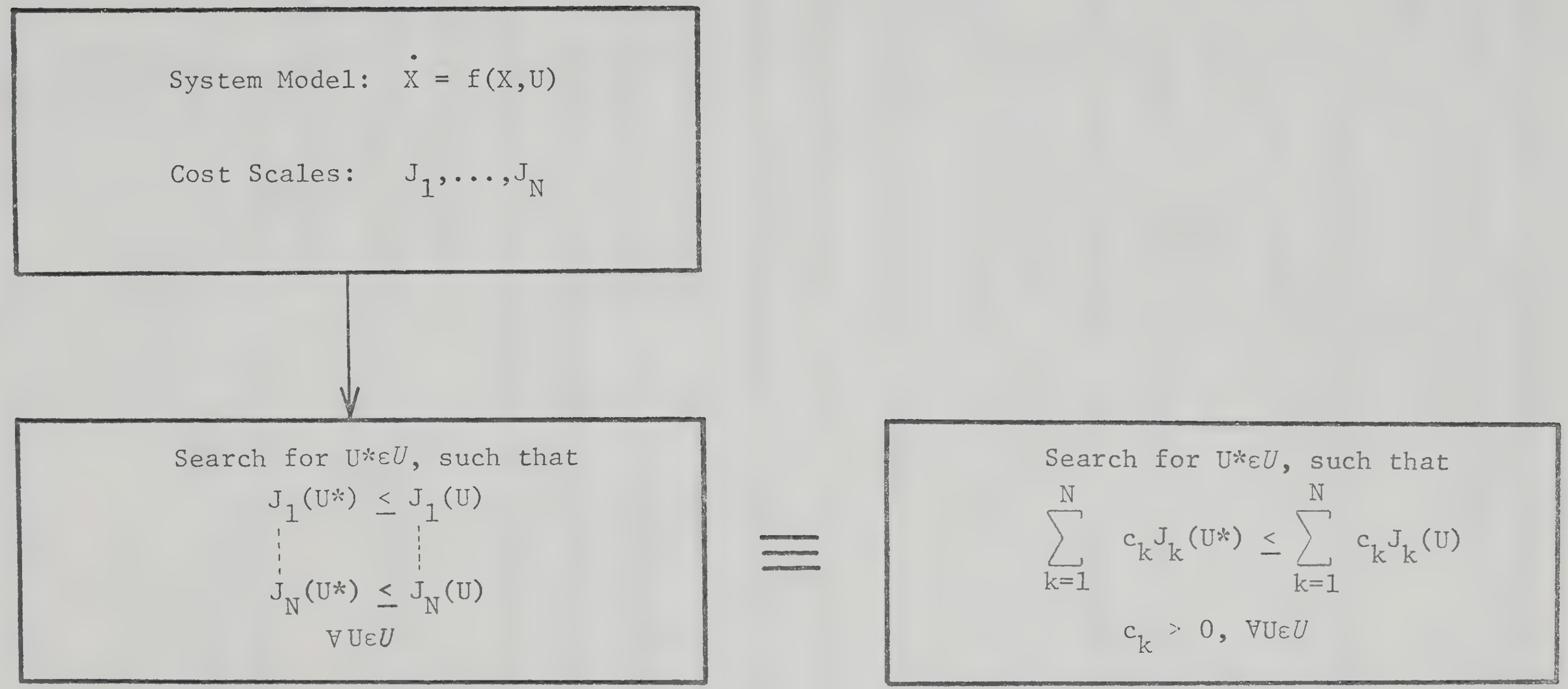


Figure (2.3) Chang's Technique

the objective function is discussed later on. The author believes that this is a more useful approach compared to the previous techniques.

In practice it may be necessary to determine the optimal control for a given system with respect to several objective functions, based on the same set of cost scales. It will be shown that there is no need to solve the optimal control problem for each objective function separately from the beginning.

Given the various cost scales J_k's the optimal control for the system is first determined with respect to an objective function $\sum_{k=1}^{N} c_k \cdot J_k$, where the c_k's are treated as parameters, (the so called linear combination problem). The c_k's may be positive or negative. The optimal control is expressed as a function of the c_k's. The specific values of the c_k's are then determined by a search technique for each objective function. This will be explained further later on.

Once the linear combination problem has been solved, the results can be used for any objective function expressed as a function of the given cost scales.

2.2 Mathematical Formulation of the Optimization Problem

Description of the Problem

Let the dynamics of the system to be controlled be represented by a set of n first order ordinary differential equations,

$$\left.\begin{aligned} &\dot{X}(t) = f[X(t), U(t)], \\ &X(t_o) = X_o \end{aligned}\right\} \quad (2.1)$$

where ($\dot{X} \equiv dX(t)/dt$). f is an n-dimensional vector valued function

continuous in X and U, and continuously differentiable with respect to X. X is an n-dimensional vector representing the state of the system. U is the control vector of the system, U has a dimension r. The control vector is required at each instant of time to satisfy certain boundedness constraints of the form,

$$\theta_j(u_1,\ldots,u_r) \leq \alpha_j, \qquad j = 1,\ldots,r \tag{2.2}$$

where α_j's are positive constants.

The control function U(t) is called an admissible control if

i) it is piecewise continuous on the control interval $[t_o,T]$, where t_o is the initial time and T is the final time, and

ii) it satisfies (2.2), $\forall t\varepsilon[t_o,T]$.

The set of admissible controls is denoted by $\mathcal{U}$. Moreover it is assumed that the state variables are required, to satisfy the following

$$\psi(X) = 0$$

at t = T. $\psi(X)$ is a real non-negative scalar function of its arguments. It is further assumed that the N cost functionals are expressible in the form,

$$J_k = g_k[X(T)] + \int_{t_o}^{T} l_k[X(\tau),U(\tau)]d\tau \leq \beta_k, \qquad k = 1,\ldots,N \tag{2.4}$$

where g_k's and l_k's are all real non-negative continuous scalar functions of their argument, the l 's are all continuously differentiable with respect to X, and the g_k's are of class C^2. The β_k's are positive constants.

Following the formulation of Knapp and Forst[6], the cost functionals in (2.4) can be put in the following form,

$$\left.\begin{aligned} &\dot{z}_k(t) = l_k[X(t),U(t)] + \left[\frac{\partial g_k[X(t)]}{\partial X}\right]' \cdot f[X(t),U(t)], \\ &z_k(t_o) = g_k[X(t_o)] \\ &k = 1,\ldots,N \end{aligned}\right\} \quad (2.5)$$

where prime denotes "transpose". It is clear that

$$J_k(T) = z_k(T), \quad k = 1,\ldots,N \qquad (2.6)$$

The $z_k(t)$'s are positive non-decreasing functions of time representing the "state of the cost scales" in the system. The reason for introducing this transformation will be clear as the discussion proceeds.

Let us define the following function

$$\phi = \phi(z_1,\ldots,z_N)$$

ϕ is of class C^2. We shall refer to ϕ as the *"Objective Function"*. In practice we associate a penalty function p_k with each cost scale z_k. This p.f. is expressed as a function of the deviation of the actual value of the cost scale and some specified desired or nominal value. In other words

$$p_k = p_k(z_k - z_{kd}), \quad k = 1,\ldots,N$$

where all the p_k's are of class C^2. These p_k's may have the dimension of dollars or some other criterion such as desirability[4]. The objective function ϕ is obtained by adding all p_k's

$$\phi = \sum_{k=1}^{N} p_k$$

Moreover, let us define the following target set B as

$$B = \{(X,Z),\ X\varepsilon\Sigma,\ \psi(X) = 0,\ z_k(T) \leq \beta_k,\ k = 1,\ldots,N\} \quad (2.7)$$

The Control Problem can now be concisely stated as follows,

Find an admissible control function $U^*(t)\varepsilon U$, *such that*

$X(T;\ X_o,\ U^*(t))\varepsilon B$

and

$$\phi[T; X_o, U^*(t)] \leq \phi[T; X_o, U(t)],$$

$$\forall U(t) \epsilon U$$

It is worth noting that, a time-dependent system can be easily transformed to a time-independent system, by introducing an extra state variable as,

$$\dot{x}_{n+1}(t) = 1, \quad x_{n+1}(t_o) = t_o$$

Therefore it is sufficient to consider the time-independent systems. The reason for introducing the transformation (2.5) is that we can cast the problem in a form for which the necessary conditions can be obtained very easily using the formulation of Rozonoer[9,10].

Necessary Conditions for Optimality

We shall hereafter write X, U, Z instead of X(t), U(t), Z(t) respectively to simplify the notation. The control problem can be described by the following system of n+N+1 "augmented state" equations,

$$\left.\begin{aligned}
&\dot{X} = f(X,U), && \text{n-equations}\\
&\dot{z}_k = l_k(X,U) + \left[\frac{\partial g_k(X)}{\partial X}\right]' f(X,U) = g_k(X,U), && \text{N-equation}\\
&\qquad\qquad k = 1,\ldots,N\\
&\dot{\phi} = \frac{\partial \phi}{\partial z_1} g_1(X,U) + \ldots + \frac{\partial \phi}{\partial z_N} g_N(X,U), && \text{1-equation}\\
&\text{and}\\
&X(t_o) = X_o\\
&z_k(t_o) = g_k[X(t_o)], \quad k = 1,\ldots,N\\
&\phi(t_o) = \phi[z_1(t_o),\ldots,z_N(t_o)]
\end{aligned}\right\} \quad (2.8)$$

assuming the final time, T, is fixed. Written in this form, this problem becomes a special case of Rozonoer's problem in which the objective function is a linear combination of the augmented states.

In the present case the objective function is one of the states, namely ϕ.

We shall follow Rozonoer's procedure to obtain the necessary conditions for optimality using Pontryagin's Maximum Principle. We shall consider the free right end point problem first. Let us define the Hamiltonian as,

$$H = P' \, f(X,U) + p_{n+1} \, g_1(X,U) + \ldots + p_{n+N} \, g_N(X,U)$$
$$+ p_{n+N+1} \left[\frac{\partial \phi}{\partial z_1} g_1(X,U) + \ldots + \frac{\partial \phi}{\partial z_N} g_N(X,U)\right] \qquad (2.9)$$

where P is an n-dimensional vector. We get,

$$\left.\begin{aligned} \dot{P} &= -\frac{\partial H}{\partial X} \\ \dot{p}_{n+k} &= -\frac{\partial H}{\partial z_k}, \qquad k = 1,\ldots,N, \text{ and} \\ \dot{p}_{n+N+1} &= -\frac{\partial H}{\partial \phi} = 0 \end{aligned}\right\} \qquad (2.10)$$

Since we are dealing with free right end point, then

$$\left.\begin{aligned} P(T) &= 0, \\ p_{n+k}(T) &= 0, \quad k = 1,\ldots,N \text{ and} \\ p_{n+N+1}(T) &= -1 \end{aligned}\right\} \qquad (2.11)$$

From $\dot{p}_{n+N+1} = 0$, we get $p_{n+N+1}(t) = -1$.

The expression for the Hamiltonian becomes,

$$H = P'.f(X,U) + (p_{n+1} - \frac{\partial \phi}{\partial z_1}) \, g_1(X,U) + \ldots\ldots$$
$$+ (p_{n+N} - \frac{\partial \phi}{\partial z_N}) \, g_N(X,U) \qquad (2.12)$$

Now let us get a close look at the quantities $(p_{n+k} - \frac{\partial \phi}{\partial z_k})$, $k = 1,\ldots,N$. That these quantities are constants, can be easily

seen from the equation of the adjoint variables $\dot{p}_{n+k}$, $k = 1,\ldots,N$.

$$\dot{p}_{n+k} = -\frac{\partial H}{\partial z_k} = \frac{\partial^2 \phi}{\partial z_1 \partial z_k} g_1(X,U) + \ldots + \frac{\partial^2 \phi}{\partial z_N \partial z_k} g_N(X,U) \quad (2.13)$$

$$k = 1,\ldots,N$$

Let

$$p_{n+k} - \frac{\partial \phi}{\partial z_k} = -c_k, \quad k = 1,\ldots,N \quad (2.14)$$

where the c_k's are constants (assuming $c_k \neq 0$, $k = 1,\ldots,N$).

Since

$$p_{n+k}(T) = 0, \quad k = 1,\ldots,N \quad (2.15)$$

therefore

$$c_k = \frac{\partial \phi}{\partial z_k}\bigg|_{t=T} \quad (2.16)$$

Before we proceed with the rest of the main derivation, we will digress briefly to comment on the c_k's. The tangent hyperplane to the surface ϕ = constant is $\sum_{k=1}^{N} c_k z_k$, at $t=T$.

The expression for the Hamiltonian becomes,

$$\left.\begin{aligned} H &= P' \, f(X,U) - \sum_{k=1}^{N} c_k g_k(X,U) \\ &= P' \, f(X,U) - \left(\sum_{k=1}^{N} c_k \left[\frac{\partial g_k(X)}{\partial X}\right]' . \, f(X,U) + \sum_{k=1}^{N} c_k l_k(X,U) \right) \end{aligned}\right\} \quad (2.17)$$

If we define

$$\tilde{P}' = P' - \sum_{k=1}^{N} c_k \left[\frac{\partial g_k(X)}{\partial X}\right]' \quad (2.18)$$

the expression for H becomes,

$$H[X(t;c_1,\ldots,c_N),\ P(t;c_1,\ldots,c_N),\ U(t;c_1,\ldots,c_N)]$$

$$= \tilde{P}' . \, f(X,U) - \sum_{k=1}^{N} c_k l_k(X,U) \quad (2.19)$$

It is clear from (2.18) and (2.10) that,

$$\dot{\tilde{P}} = -\frac{\partial H}{\partial X}$$

$$\tilde{P}(T) = -\sum_{k=1}^{N} c_k\left[\frac{\partial g_k(X)}{\partial X}\right]\Big|_{t=T} \tag{2.20}$$

The preceding results can be summarized in the form of a theorem.

Theorem (1) A necessary condition for optimality of U**(t)*εU *in system (2.8) is that the Hamiltonian defined in (2.19) attains an absolute maximum for every* τ, $t_o \leq \tau \leq T$. *This optimal control will be a function of the* c_k*'s, defined by (2.16).*

For the following system

$$X(t) = A(t)X(t) + \Omega[U(t)] \tag{2.21}$$

where A(t) is nxn matrix and Ω is a vector-valued function of dimensionality n, the following holds

Theorem (2) A necessary and sufficient condition for optimality of U**(t)*εU *in system (2.21) is that the corresponding Hamiltonian attains an absolute maximum for every* τ, $t_o \leq \tau \leq T$.

Turning now to the problem of constrained end point; assume that the final state X(T) satisfies the condition

$$\psi[X(T)] = 0$$

Then the boundary conditions in (2.20), have to be modified to

$$\tilde{P}(T) = -\sum_{k=1}^{N} c_k\left[\frac{\partial g_k(X)}{\partial X}\right]\Big|_{t=T} - \tilde{\alpha}\,\frac{\partial \psi(X)}{\partial X}\Big|_{t=T} \tag{2.22}$$

where $\tilde{\alpha}$ is a positive constant.

It T is not fixed this will require that,

$$H[X(t;c_1,\ldots,c_N),\ P(t;c_1,\ldots,c_N),\ U(t;c_1,\ldots,c_N)] = 0 \tag{2.23}$$

at t=T. Moreover if some of the state variables are fixed at t=T the corresponding adjoint variables are not specified at t=T, this may be applicable to the cost scales z_k's too.

2.3 A Search Technique for Determining the Optimum Control

We have already pointed out, for the case where there are no constraints on the cost scales, that as a consequence of the condition for optimality,

$$c_k^* = \left.\frac{\partial \phi}{\partial z_k}\right|_{t=T} \neq 0 \qquad (2.16)$$

We find that for a given objective function ϕ, its optimal value at t=T occurs at the same point in the Z-space as the optimum value of $\sum_{k=1}^{N} c_k^* z_k$ at t=T. Since it is easier to handle the optimization problem in which the objective function is a linear combination of the cost scales, the optimal control for the problem under discussion can be determined in the following manner.

Step (1) Consider the optimization problem with the same system equations and same boundary conditions as for the given problem but with objective function $\sum_{k=1}^{N} c_k z_k$, where the c_k's are parameters. The details involved in determining the optimal control U* as function of the c_k's are best explained by means of a numerical example. Therefore further discussion of this step will be postponed to a later section. At this stage, for the purpose of explaining step (2), we will assume that U* has been determined as a function of the c_k's.

Step (2) This step is concerned with the search of the optimum values of the c_k's which minimize the given ϕ. If there are direct

relationships between the various cost scales then this search can be carried out in the Z-space. Otherwise the search will have to be carried out in the C-space. There may arise situations where a modification of these techniques will have to be used. These points will be clarified further by the numerical examples which may be found in a later section. At this stage it will be possible only to describe the basic concepts involved. A flow chart describing this procedure is shown in figure (2.4). In what follows we are going to consider the case where c_k's are all positive. The extension to cases where we have +ve and -ve c_k's is straightforward. Let us denote the set of points in the Z-space corresponding to all possible variations in c_j/c_1, $j = 2,\ldots,N$, at $t=T$ by π.

Z-Space Search

In this search direct relations must exist between the various cost scales in terms of the ratios c_j/c_1, $j = 2,\ldots,N$, otherwise we use the C-space search. For the Z-space search, let us pick one of the cost scales as a dependent variables, say z_1, and the remaining N-1 scales as independent variable. The expression of the gradient of ϕ is expressed as,

$$[\text{grad } \phi] = \begin{bmatrix} \partial\phi/\partial z_2 \\ \cdot \\ \cdot \\ \partial\phi/\partial z_N \end{bmatrix} = \begin{bmatrix} (\partial\phi/\partial z_1)(\partial z_1/\partial z_2) + (\partial\phi/\partial z_2) \\ \cdot \\ \cdot \\ (\partial\phi/\partial z_1)(\partial z_1/\partial z_N) + (\partial\phi/\partial z_N) \end{bmatrix} \tag{2.24}$$

At any point $Z\varepsilon\pi$, the tangent hyperplane to π is described by

$$\sum_{k=1}^{N} c_k z_k = \text{constant} \tag{2.25}$$

For the objective function $\sum_{k=1}^{N} c_k z_k$

determine $U^* = U^*(c_1,\ldots,c_N)$

$\downarrow$

Search in the Z-Space or C-Space for

the ratios c_j/c_1, $j = 2,\ldots,N$ which minimize ϕ

$\downarrow$

Determine the optimal control

$U^*(t)$ and carry out sensitivity analysis

of ϕ with respect to c_j/c_1's

Figure (2.4) Proposed Optimization Technique

Differentiating (2.25) successively with respect to z_j's, $j = 2,\ldots,N$, we get

$$\partial z_1/\partial z_j = -c_j/c_1, \quad j = 2,\ldots,N \tag{2.26}$$

Substituting (2.26) into (2.24),

$$[\text{grad } \phi] = \begin{bmatrix} (\partial\phi/\partial z_1)(-c_2/c_1) + (\partial\phi/\partial z_2) \\ \vdots \\ (\partial\phi/\partial z_1)(-c_N/c_1) + (\partial\phi/\partial z_N) \end{bmatrix} \tag{2.27}$$

All the components of the gradient vector are known at any point in π. It must be noted that if the objective function ϕ attains its minimum at $Z^*\varepsilon\pi$, then the components of the gradient vector are zeros. Therefore, setting equation (2.27) = 0, we get

$$(\partial\phi/\partial z_j)/(\partial\phi/\partial z_1) = c_j^*/c_1^* \quad j = 2,\ldots,N \tag{2.28}$$

It is clear that these are the same as the conditions given in (2.16). The problem now reduces to the determination of the minimum of a function assuming that its value and its gradient at each point of the search region are known. In this regard many techniques grouped under the broad category of steepest descent methods are available. The methods used in the examples of this chapter are the "Bi-Section Search" for one dimensional search and the "Accelerating-Step Search" [8], for the 2-dimensional search.

C-Space Search

If there are no direct relationships between cost scales in terms of c_j/c_1, $J = 2,\ldots,N$, the previous method cannot be used. Then we use C-space search. In this search the components of the gradient of ϕ are given by,

$$[\text{grad } \phi] = \begin{bmatrix} (\partial\phi/\partial z_1)(\partial z_1/\partial(c_2/c_1)) + \ldots + (\partial\phi/\partial z_N)(\partial z_N/\partial(c_2/c_1)) \\ \vdots \qquad\qquad\qquad \vdots \\ (\partial\phi/\partial z_1)(\partial z_1/\partial(c_N/c_1)) + \ldots + (\partial\phi/\partial z_N)(\partial z_N/\partial(c_N/c_1)) \end{bmatrix} \tag{2.29}$$

All the components of the gradient vector are known if c_j/c_1, $j = 2,\ldots,N$, are specified. This is the case for each point Z in π. The gradient technique can be used to determine optimum values of $(c_j/c_1)^*$, $j = 2,\ldots,N$, which minimize the given ϕ.

If the number of the cost scales becomes very large, the gradient technique becomes unwieldy. We can use what is known as the "Random Search" technique[2]. The use of the techniques involving Z-space search and C-space search will be illustrated by the examples that follow.

2.4 Examples

The purpose of this section is to illustrate the application of the theory and computational technique presented earlier in this chapter. Three examples will be discussed. The first example will deal with a system with two cost scales. The second will consider the same system with three cost scales. In these two examples the use of the Z-space search technique will be demonstrated. The concluding example is intended to illustrate the use of the C-space search technique.

The single-axis satellite attitude control problem is one of the most popular examples used to illustrate optimal control techniques. Since it is a good illustration for multi-cost control problems, it will be discussed in all examples in this chapter. The equation of

motion of the satellite about its centre of gravity, under the influence of gas-jet controllers with limited thrust, is given by

$$I \frac{d^2\theta(t)}{dt^2} = L(t), \qquad |L(t)| \leq M \tag{2.30}$$

where $\theta(t)$ is the attitude error angle, I is the moment of inertia, and $L(t)$ is the control torque limited by the maximum magnitude M. Defining the normalized variables,

$$\left.\begin{aligned} x_1(t) &\equiv \frac{I}{M}\,\theta(t), \\ x_2(t) &\equiv \frac{I}{M}\,\dot{\theta}(t), \text{ and} \\ u(t) &\equiv L(t)/M \end{aligned}\right\} \tag{2.31}$$

the equations of motion and control constraints are written in the form,

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t), & x_1(t_o) &= x_{10} \\ \dot{x}_2(t) &= u(t), & x_2(t_o) &= x_{20} \\ |u(t)| &\leq 1 & \text{and} \quad t &\varepsilon[t_o,T] \end{aligned}\right\} \tag{2.32}$$

where T is the final time.

Example (1)

Consider a system with dynamics given by equation (2.32) and the following two cost functionals,

$$\left.\begin{aligned} J_1 &= \int_{t_o}^{T} d\tau \quad , \text{ and} \\ J_2 &= \int_{t_o}^{T} |u(\tau)|\, d\tau \end{aligned}\right\} \tag{2.33}$$

where J_1 is the time elapsed and J_2 is the fuel consumed. The objective is to reduce the attitude and its derivative to zero while minimizing

$$\phi(J_1,J_2)\Big|_{t=T} \tag{2.34}$$

where ϕ is some specified objective function.

Following the discussion in section 2.2, the augmented state equations are,

$$\left.\begin{aligned}
\dot{x}_1(t) &= x_2(t), & x_1(t_o) &= x_{10}\\
\dot{x}_2(t) &= u(t), & x_2(t_o) &= x_{20}\\
\dot{z}_1(t) &= 1, & z_1(t_o) &= 0\\
\dot{z}_2(t) &= |u(t)|, & z_2(t_o) &= 0
\end{aligned}\right\} \quad (2.35)$$

and

$$|u(t)| \leq 1 \qquad t\varepsilon[t_o,T]$$

T is free, and $x_1(T) = x_2(T) = 0$

Since the given system is time invariant, we can set $t_o = 0$.

In this example two objective functions will be used,

$$\text{(i)} \quad \phi(z_1,z_2) = z_1{}^r z_2 \quad \text{where } r > 1 \tag{2.36}$$

$$\text{(ii)} \quad \phi(z_1,z_2) = az_1{}^2+z_2 \quad \text{where } a > 0 \tag{2.37}$$

Step (1) Solution to the Linear Combination Problem

This step is common to both the objective functions. Consider the objective function $c_1z_1 + c_2z_2$. The Hamiltonian is given by

$$H = p_1x_2 + p_2u-c_1-c_2|u| \tag{2.38}$$

And

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = 0, \quad p_1(T) \text{ undefined since } x_1(T) \text{ is fixed} \tag{2.39}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -p_1 \quad p_2(T) \text{ undefined since } x_2(T) \text{ is fixed} \tag{2.40}$$

We shall first begin by assuming that $x_{20} = 0$, because it leads to a symmetric control law. Moreover the Hamiltonian is identically zero with respect to time, since we are dealing with a time invariant system. It is obvious that $T = z_1(T)$. Let us designate $z_2(T)$ by F. Then after some manipulations we arrive at the expression

$$u^*(t) = \mathrm{DEZ}[(1+c_1/c_2)(2t/T-1)], \ \forall t \varepsilon [0, T] \quad (2.41)$$

where DEZ means "Dead Zone Function" Figure (2.5)[1]. We also get

$$T = \frac{2(1+c_2/c_1)}{\sqrt{1+2(c_2/c_1)}} \sqrt{x_{10}}, \qquad x_{20} = 0 \quad (2.42)$$

$$F = \frac{2}{\sqrt{1+2(c_2/c_1)}} \sqrt{x_{10}} \qquad x_{20} = 0 \quad (2.43)$$

It is worthnoting that if $(c_2/c_1) = 0$, we get

$$T = F = 2\sqrt{x_{10}} \ , \qquad x_{20} = 0 \quad (2.44)$$

which is the case of minimum time. If $c_1 \to 0$, we get

$$\left.\begin{array}{l} T \to \infty \\ F \to 0 \end{array}\right\} , \qquad x_{20} = 0 \quad (2.45)$$

The equations of the switching curves in the x_1x_2 space are given by

$$\left.\begin{array}{l} x_1 = (0.5+2(c_2/c_1))\ x_2|x_2| \ , \\ \text{and} \\ x_1 = 0.5\ x_2\ |x_2| \end{array}\right\} \quad (2.46)$$

These switching curves are shown in figure (2.6). These switching curves are obviously valid for initial states (x_{10}, x_{20}) with $x_{20} \neq 0$.

The expressions for the time elapsed to reach the state (0,0) from any initial state $(x_{10},x_{20})\varepsilon G^-$ and $(x_{10},x_{20})\varepsilon G^+$ and the corresponding fuel consumed as functions of c_2/c_1 and the initial state are given by

$$\left.\begin{array}{l} T = \pm\ x_{20} + 2(1+c_2/c_1)\sqrt{(2x_{10}+x_{20}^2)/(2+4(c_2/c_1))} \\ F = \pm\ x_{20} + 2\sqrt{(2x_{10}+x_{20}^2)/(2+4(c_2/c_1))} \end{array}\right\} \quad (2.47)$$

where the +ve sign is for all the initial state $(x_{10},x_{20})\varepsilon G^-$, and the -ve sign is for all the initial states $(x_{10},x_{20})\varepsilon G^+$, figure (2.6). As

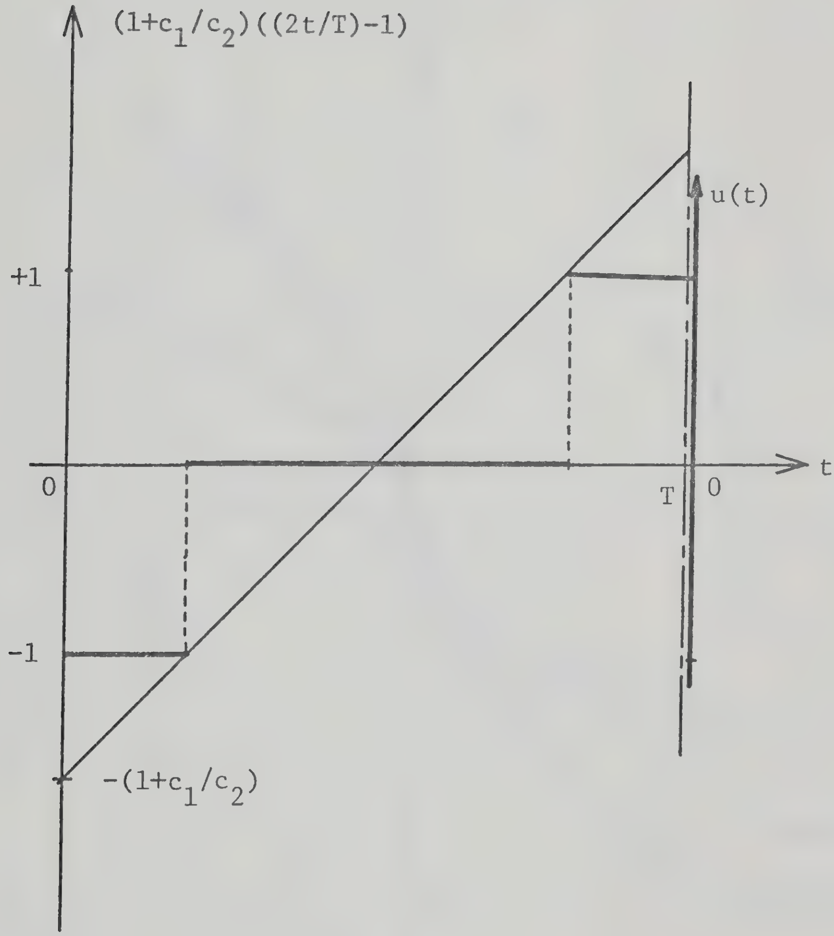


Figure (2.5) Optimal Control for Example (1) - (Chapter (2))

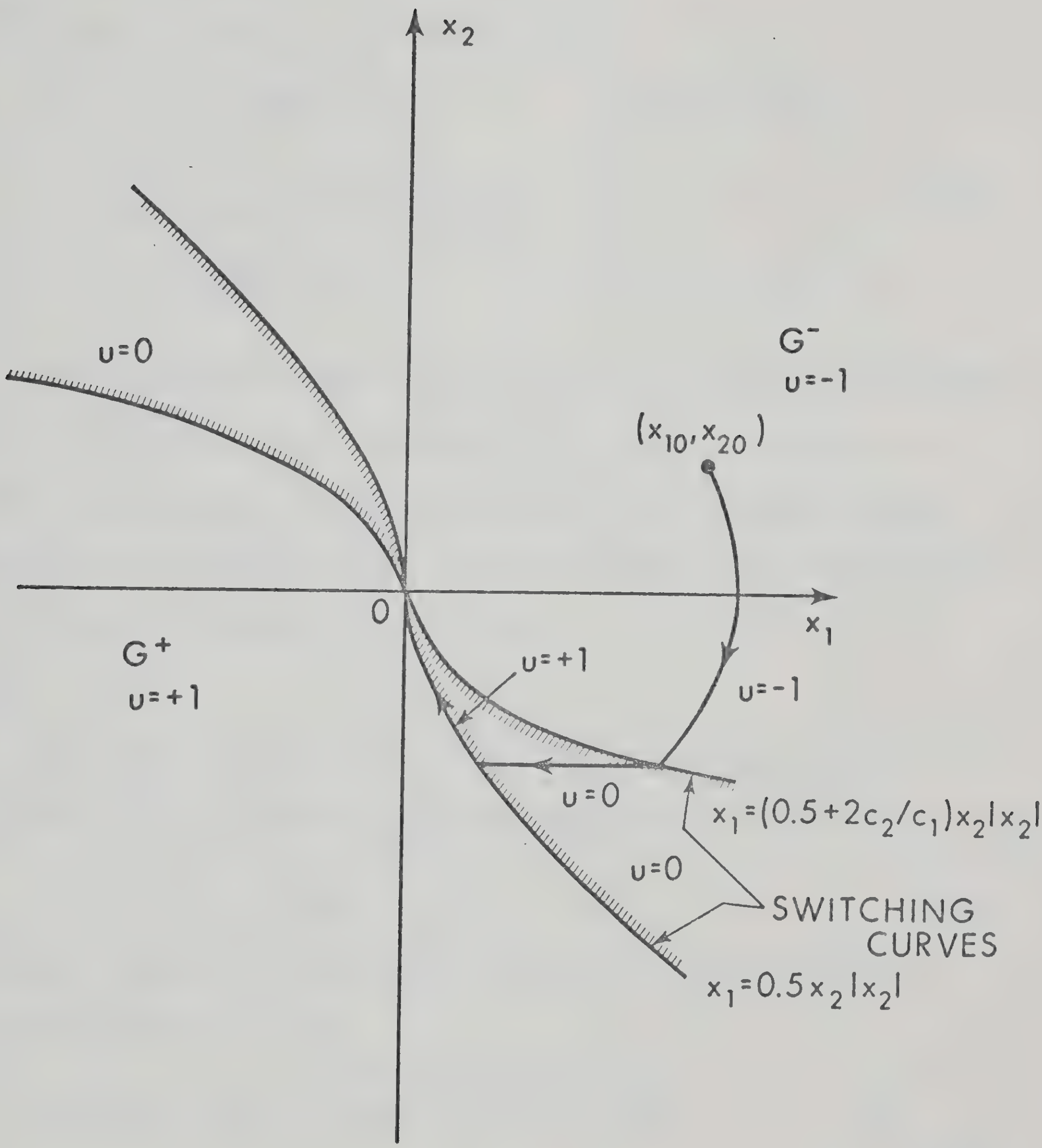


Figure (2.6) Switching Curves for Example (1) (Chapter (2))

typical initial conditions, let $x_{10} = x_{20} = 1$, and $x_{10} = 1$, $x_{20} = 0$. For these cases equation (2.47) becomes

$$\left.\begin{aligned} &\left.\begin{aligned} T &= 1 + 2(1+c_2/c_1)\sqrt{3/(2+4c_2/c_1)} \\ F &= 1 + 2\sqrt{3/(2+4c_2/c_1)} \end{aligned}\right\} (x_{10}=1,\ x_{20}=1) \\ &\left.\begin{aligned} T &= 2(1+c_2/c_1)\sqrt{2/(2+4c_2/c_1)} \\ F &= 2\sqrt{2/(2+4c_2/c_1)} \end{aligned}\right\} (x_{10}=1,\ x_{20}=0) \end{aligned}\right\} \quad (2.48)$$

The relation between F and T, as c_2/c_1 varies, for the given initial conditions is given in figure (2.7). These relations are identical with the ones obtained by Nelson[7] for the given initial conditions using another method. The realization of the control law given in (2.41) is shown in figure (2.8).

Step (2)

We shall now proceed to determine the optimum value of c_2/c_1, which will minimize the objective function

$$\phi = z_1^r z_2 , \quad \text{where } r > 1 \quad (2.49)$$

The two cost scales $z_1(T)$ and $z_2(T)$ are directly related because, from equation (2.47), we get,

$$\left.\begin{aligned} T &= F + (c_2/c_1)\sqrt{(2x_{10}+x_{20}^2)/(0.5+c_2/c_1)} \\ &\text{or} \\ z_1 &= z_2 + (c_2/c_1)\sqrt{(2x_{10}+x_{20}^2)/(0.5+c_2/c_1)} \end{aligned}\right\} \quad (2.50)$$

We can use the Z-space search technique described in section (2.3). z_2 is taken as dependent variable and z_1 as the independent variable. The "Bisection Search" technique will be used to solve this one dimensional search problem.

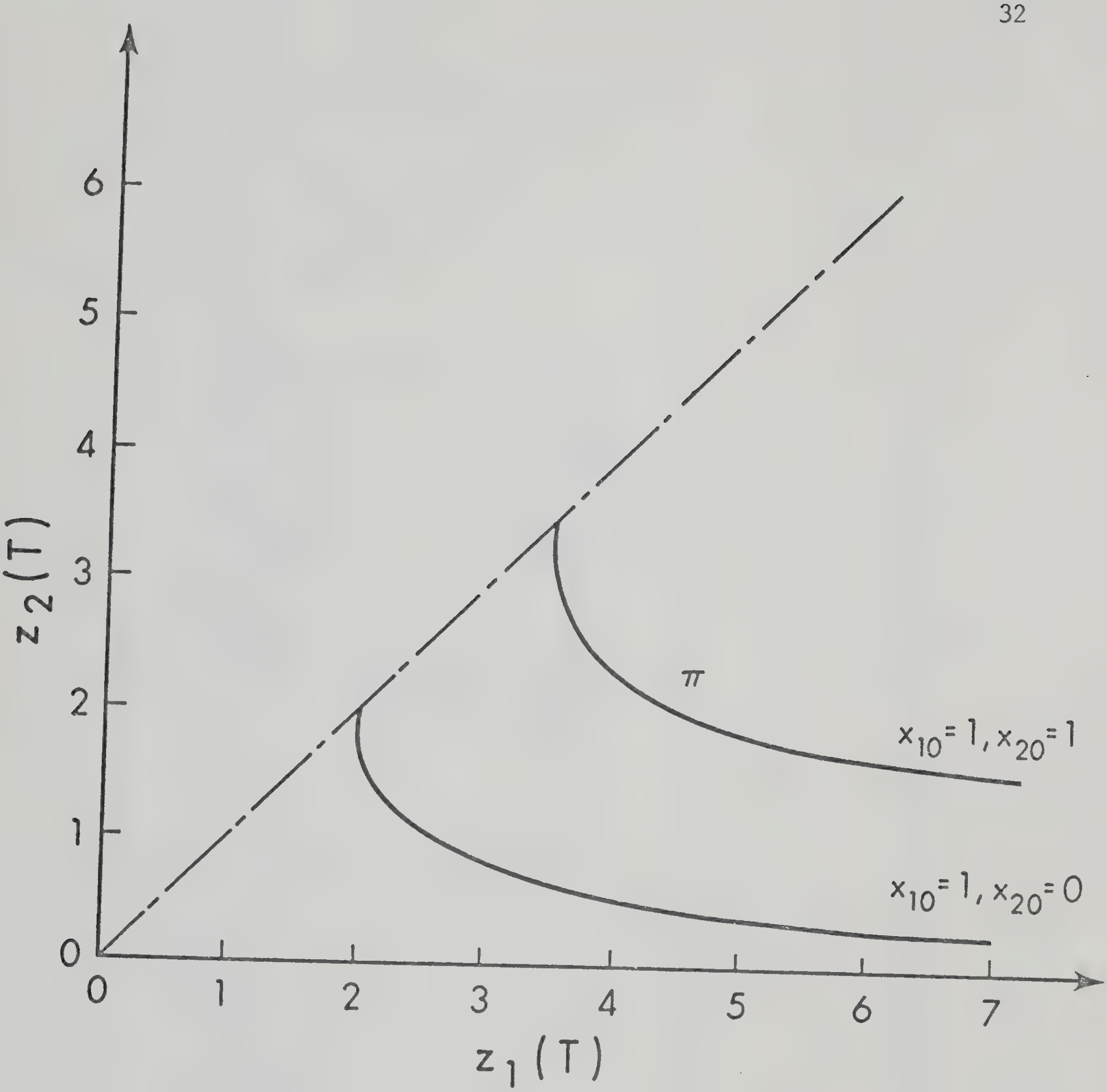
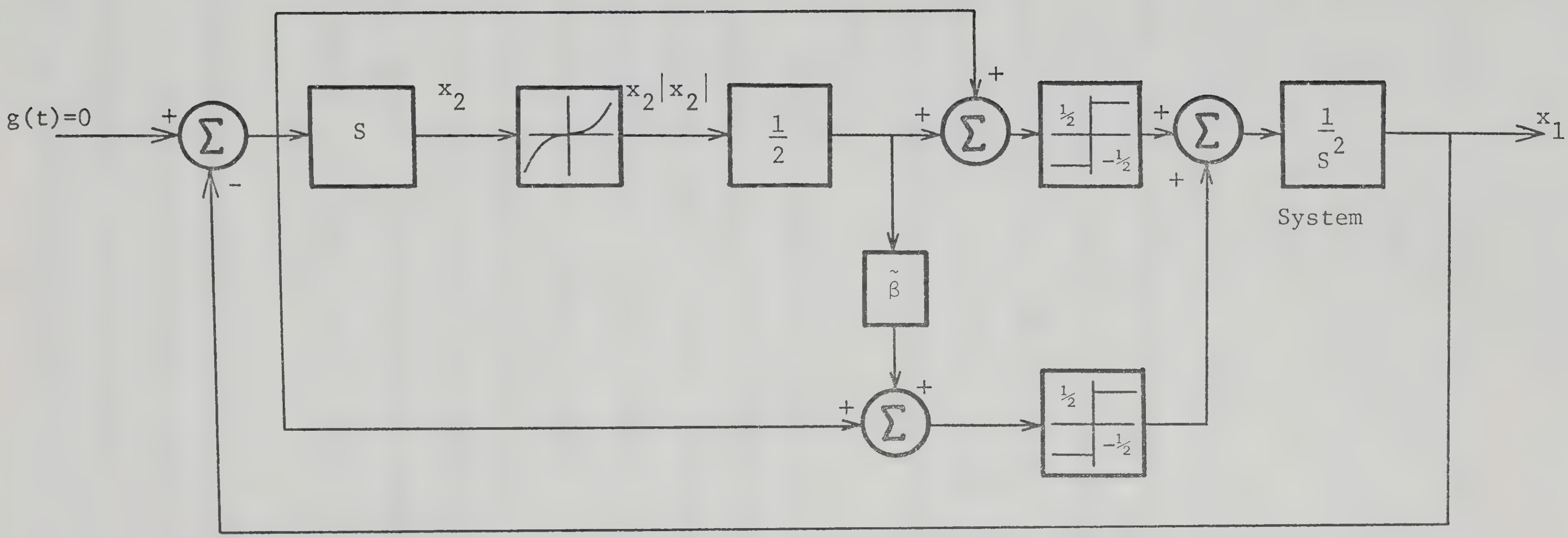


Figure (2.7) Relationship between $z_1(T)$ and $z_2(T)$ in Example (1) (Chapter (2))



$\tilde{\beta} = 1 + 4c_2/c_1$

Figure (2.8) Realization of Control Law of Example (1) (Chapter (2))

From this point, the search technique will be carried out for $x_{10} = x_{20} = 1$ and several values of r. The initial values for z_1 and z_2 are taken to correspond to $c_2/c_1 = 0$. From equation (2.48), setting $c_2/c_1 = 0$, we get,

$$z_1 = z_2 = 1 + \sqrt{6} \qquad \text{when } c_2/c_1 = 0 \tag{2.51}$$

By assuming a reasonable range of values of z_1, the minimum of $\phi(z_1,z_2)$ is located. Table (2.1) shows the optimum values of c_2/c_1 for different values or r. The sensitivity of ϕ with respect to variations in (c_2/c_1) for different values of r is shown in figure (2.9). The variation of $(c_2/c_1)^*$ with respect to r is shown in figure (2.10).

Before proceeding to consider the second objective function, it is interesting to compare the results obtained for the first objective function with the results of Kalinin[5] who also used the same objective function. It can be seen from table (2.2), that our results are identical to those of Kalinin's method for initial states on the x_1-axis. However, for initial states not on the x_1-axis, our values of ϕ are smaller than those obtained by using Kalinin's method (see table (2.3)). It would therefore appear that the generalization made by Kalinin to include initial states not on the x_1-axis is not valid.

Minimization of the Second Objective Function

We shall now consider the objective function $\phi = az_1^2+z_2$. As pointed out earlier, it is not necessary to repeat step (1). Even in step (2), only some algebraic computations are needed. The results

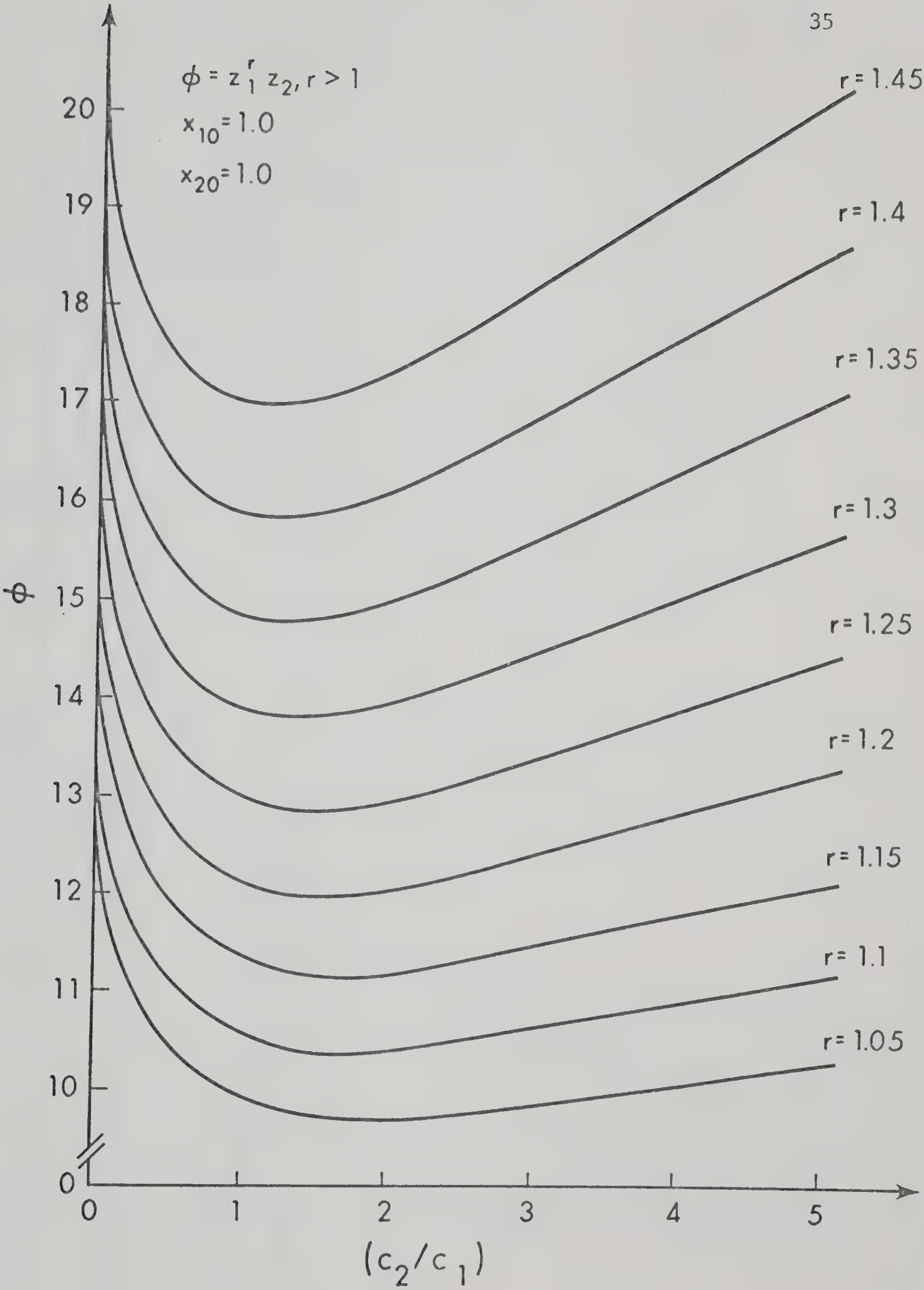


Figure (2.9) Variation of $\phi=z_1^r z_2$ with respect to (c_2/c_1)

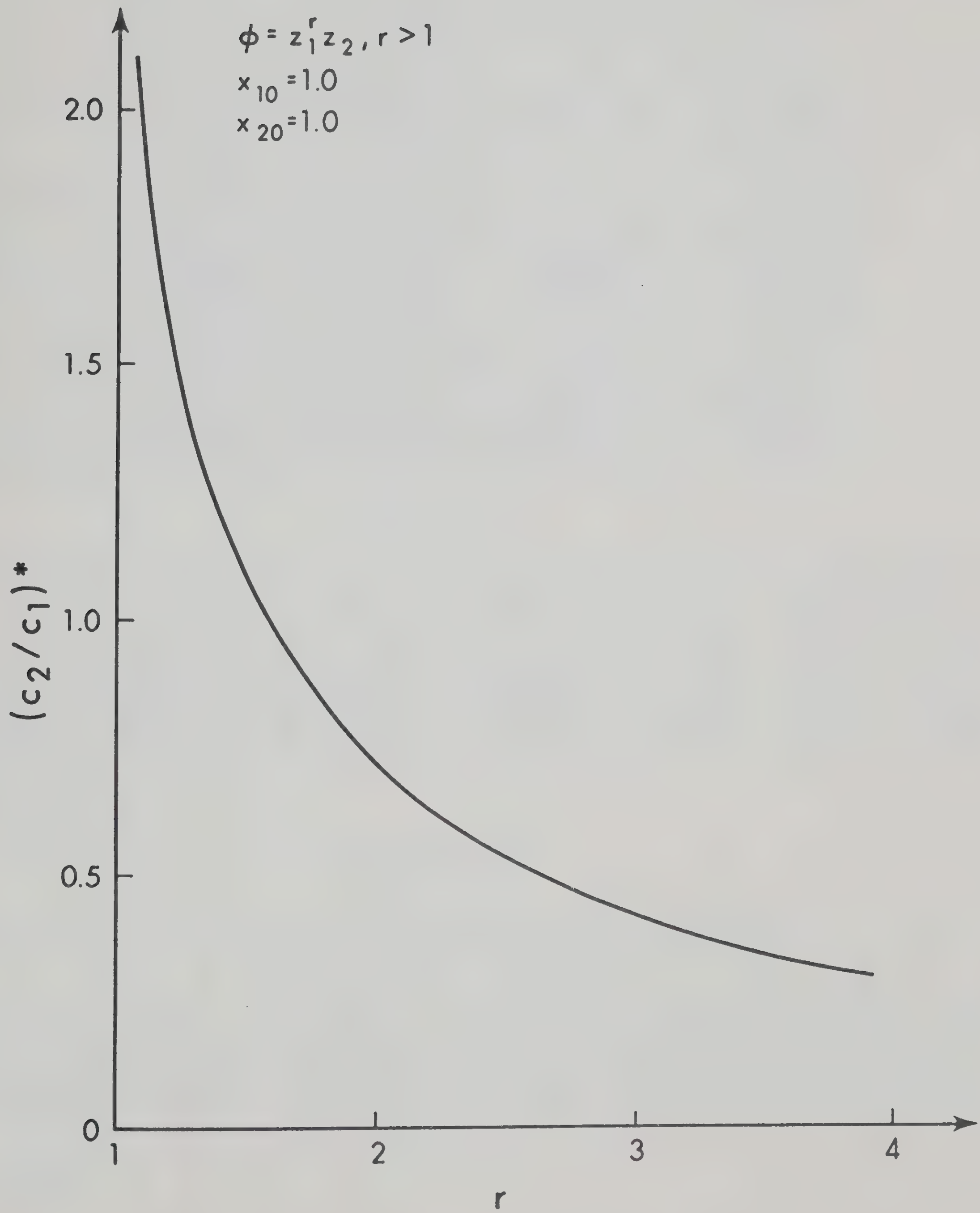


Figure (2.10) Variation of $(c_2/c_1)^*$ with respect to r for $\phi=z_1^r z_2$

r	z_1*	z_2*	ϕ*	(c_2/c_1)*
1.05	4.24795283	2.11506802	9.65857396	1.91278448
1.10	4.18412274	2.14975856	10.37897687	1.76938381
1.15	4.12776554	2.18279323	11.14512319	1.64438910
1.20	4.07775166	2.21427231	11.96013157	1.53464699
1.25	4.03315727	2.24429086	12.82730992	1.43765936
1.30	3.99322088	2.27293857	13.75016884	1.35142603
1.35	3.95731046	2.30029955	14.73243591	1.27433014
1.40	3.92489797	2.32645243	15.77807069	1.20505304
1.45	3.89553943	2.35147059	16.89128064	1.14251013

Table (2.1)

r	1.2	1.25	1.3	1.4	1.45
ϕ* Kalinin's Method	2.8217	3.0027	3.1838	3.5513	3.7394
ϕ* Proposed Technique	2.8217	3.0027	3.1838	3.5513	3.7394

$x_{10}=1.0$
$x_{20}=0.0$

Table (2.2)

r	1.2	1.25	1.3	1.4	1.45
ϕ* Kalinin's Method	13.2456	13.8621	14.5935	16.4258	17.4487
ϕ* Proposed Technique	11.9601	12.8273	13.7502	15.7781	16.8913

$x_{10}=1.0$
$x_{20}=1.0$

Table (2.3)

for this objective function are shown in Table (2.4), and the sensitivity analysis is shown in figure (2.11). The variation of $(c_2/c_1)^*$, with respect to a is shown in figure (2.12).

a	z_1^*	z_2^*	ϕ^*	$(c_2/c_1)^*$
0.1	3.95290274	2.30377127	3.86631528	1.26489325
0.2	3.67855294	2.59474214	5.30109249	0.67961507
0.3	3.58312168	2.76304997	6.61467826	0.46514376
0.4	3.53750881	2.87497508	7.88056250	0.35335602
0.5	3.51196190	2.95522806	0.12216624	0.28474113

Table (2.4)

Example (2)

Consider a system with dynamics given by equation (2.32) and the following three cost functionals,

$$\left.\begin{aligned} J_1 &= \int_0^T d\tau \\ J_2 &= \int_0^T |u(\tau)|\,d\tau, \text{ and} \\ J_3 &= \int_0^T u^2(\tau)\,d\tau \end{aligned}\right\} \quad (2.52)$$

where J_1 is the time elapsed, J_2 is the fuel consumed, and J_3 is the energy dissipated. The objective is to reduce the attitude and its derivative to zero while minimizing

$$\phi(J_1, J_2, J_3)\Big|_{t=T}$$

where ϕ is some specified objective function.

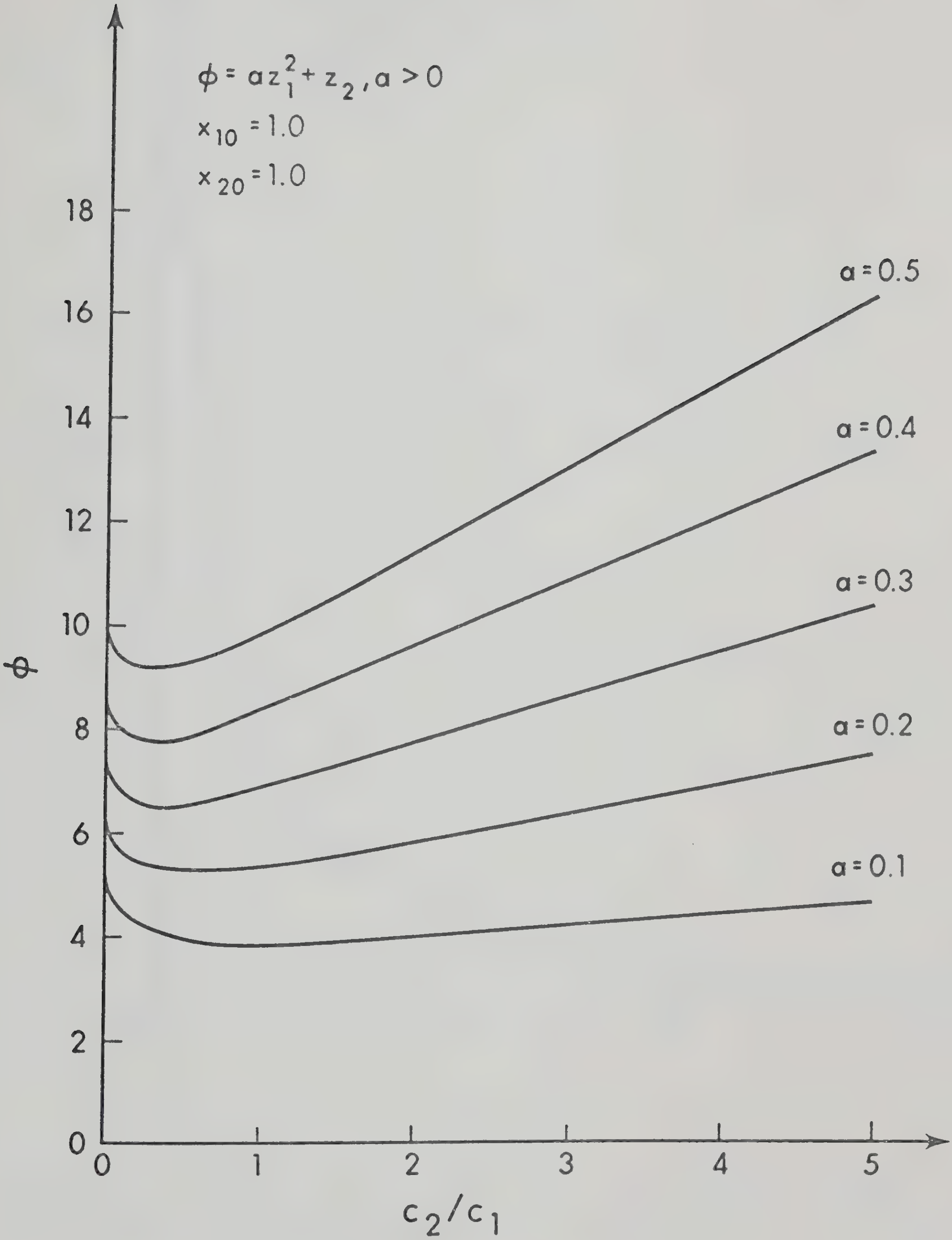


Figure (2.11) Variation of $\phi=az_1^2+z_2$ with respect to (c_2/c_1)

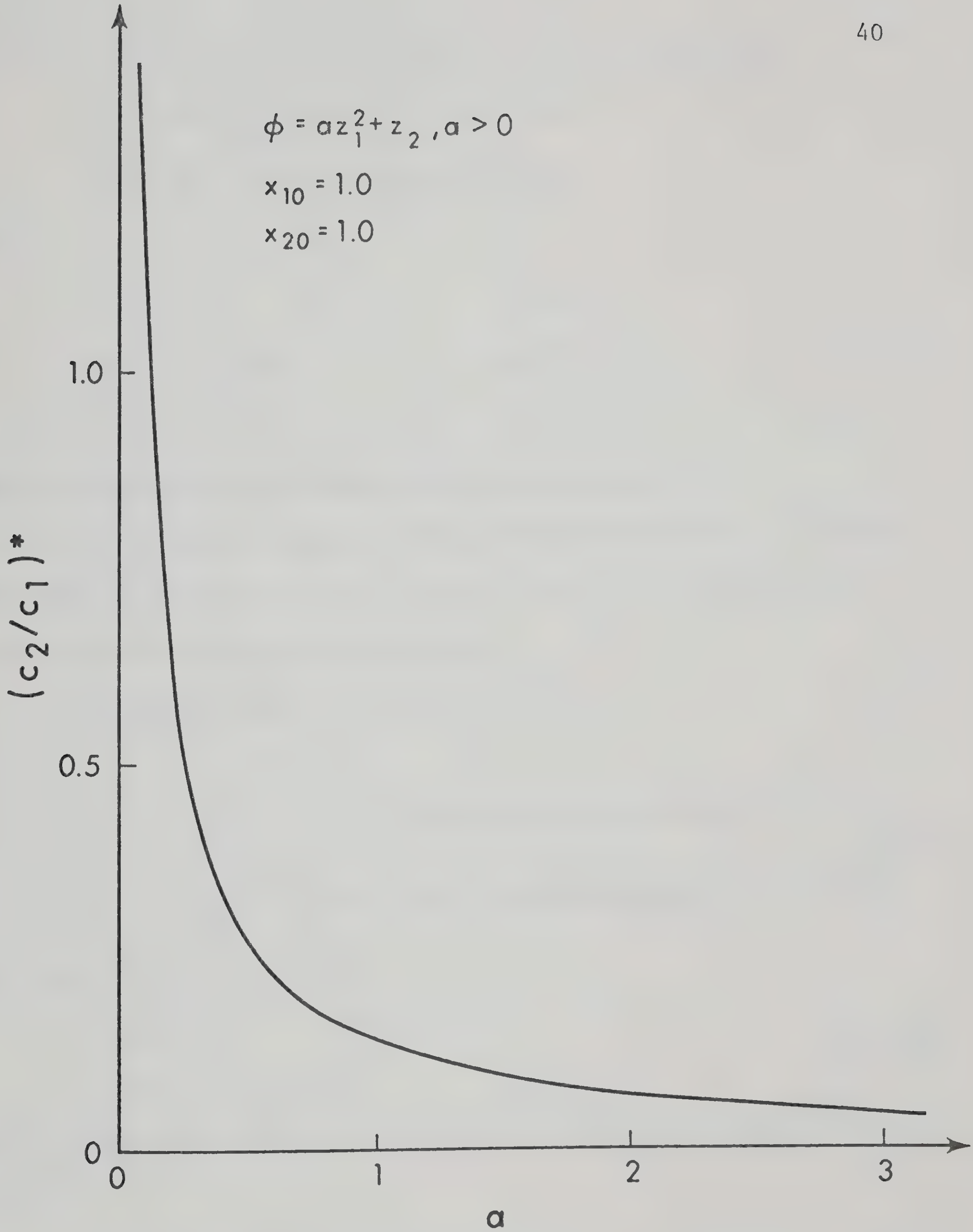


Figure (2.12) Variation of $(c_2/c_1)^*$ with respect to 'a' for $\phi = az_1^2 + z_2$

In this example two objective functions will be considered,

(i) $\phi(z_1,z_2,z_3) = z_1^2+z_2^2+z_3^2$, and

(ii) $\phi(z_1,z_2,z_3) = 10(z_1-2)^2+z_2+z_3$

where

$$\left.\begin{aligned} \dot{z}_1(t) &= 1 & z_1(0) &= 0 \\ \dot{z}_2(t) &= |u(t)| & z_2(0) &= 0 \\ \dot{z}_3(t) &= u^2(t) & z_3(0) &= 0 \end{aligned}\right\} \quad (2.53)$$

Step (1) Solution to the Linear Combination Problem

This step is common to both objective functions. Consider the objective function $\phi = c_1z_1 + c_2z_2 + c_3z_3$, for this objective function the Hamiltonian takes the form

$$H = p_1x_2 + p_2u - c_1 - c_2|u| - c_3u^2 \quad (2.54)$$

where

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial H}{\partial x_1} = 0 & &p_1(T) \text{ undefined since } x_1(T) \text{ is fixed} \\ & & &\qquad\qquad\qquad\qquad (2.55) \\ \dot{p}_2 &= -\frac{\partial H}{\partial x_2} = -p_1(t) & &p_2(T) \text{ undefined since } x_2(T) \text{ is fixed} \end{aligned}$$

From equation (2.55), we get

$$\left.\begin{aligned} p_1(t) &= \omega_1 \\ p_2(t) &= -\omega_1t + \omega_2 \end{aligned}\right\} \quad (2.56)$$

where ω_1 and ω_2 are unknown constants. The expression for H in (2.54) becomes

$$\begin{aligned} H &= \tilde{\omega}_1x_2 + (-\tilde{\omega}_1t + \tilde{\omega}_2)u-1 - (c_2/c_1)\ u\ - (c_3/c_1)u^2 \quad (2.57) \\ &= 0, \quad \forall t\varepsilon[0,T] \end{aligned}$$

where

$\tilde{\omega}_1 = \omega_1/c_1$, and

$\tilde{\omega}_2 = \omega_2/c_1$

The control $u^*(t)$ which maximizes H is given by

$$u^*(t) = SAD[-\tilde{\omega}_1 t + \tilde{\omega}_2;\ \eta_1,\eta_2] \tag{2.58}$$

where

$$\eta_1 = c_2/c_1, \text{ and}$$

$$\eta_2 = 2(c_3/c_1) + (c_2/c_1)$$

SAD means "Saturation-Dead-Zone" function, see figure (2.13). For a derivation of (2.58) see the Appendix at the end of this chapter.

We are going to consider first the case where $x_{10} > 0$ and $x_{20} = 0$. Moreover from equation (2.57) the Hamiltonian is identically zero $\forall\, t \varepsilon [0,T]$. Then from $H_{t=0} = 0$, we get

$$\tilde{\omega}_2 = -(1 + c_2/c_1 + c_3/c_1) \tag{2.59}$$

providing that $u(0)$ is -1. Similarly for $H_{t=T} = 0$, we have

$$\tilde{\omega}_1 = -(2/T)(1 + c_2/c_1 + c_3/c_1) \tag{2.60}$$

providing that $u(T) = +1$. The expression of the control law given by equation (2.58) becomes, after substituting equations (2.59) and (2.60),

$$u^*(t) = SAD\{[(1 + c_2/c_1 + c_3/c_1)(2t/T - 1)];\ \eta_1,\eta_2\} \tag{2.61}$$

(2.61)

From equation (2.61) we can derive expressions for τ_1, τ_2, and τ_3, see figure (2.13),

at $t = \tau_1$,

$$-\tilde{\omega}_1\tau_1 - (1 + c_2/c_1 + c_3/c_1) = -(c_2/c_1 + 2c_3/c_1)$$

We get

$$\tau_1/T = 0.5(1 - c_3/c_1)/(1 + c_2/c_1 + c_3/c_1) \tag{2.62}$$

And at $t = \tau_1 + \tau_2$

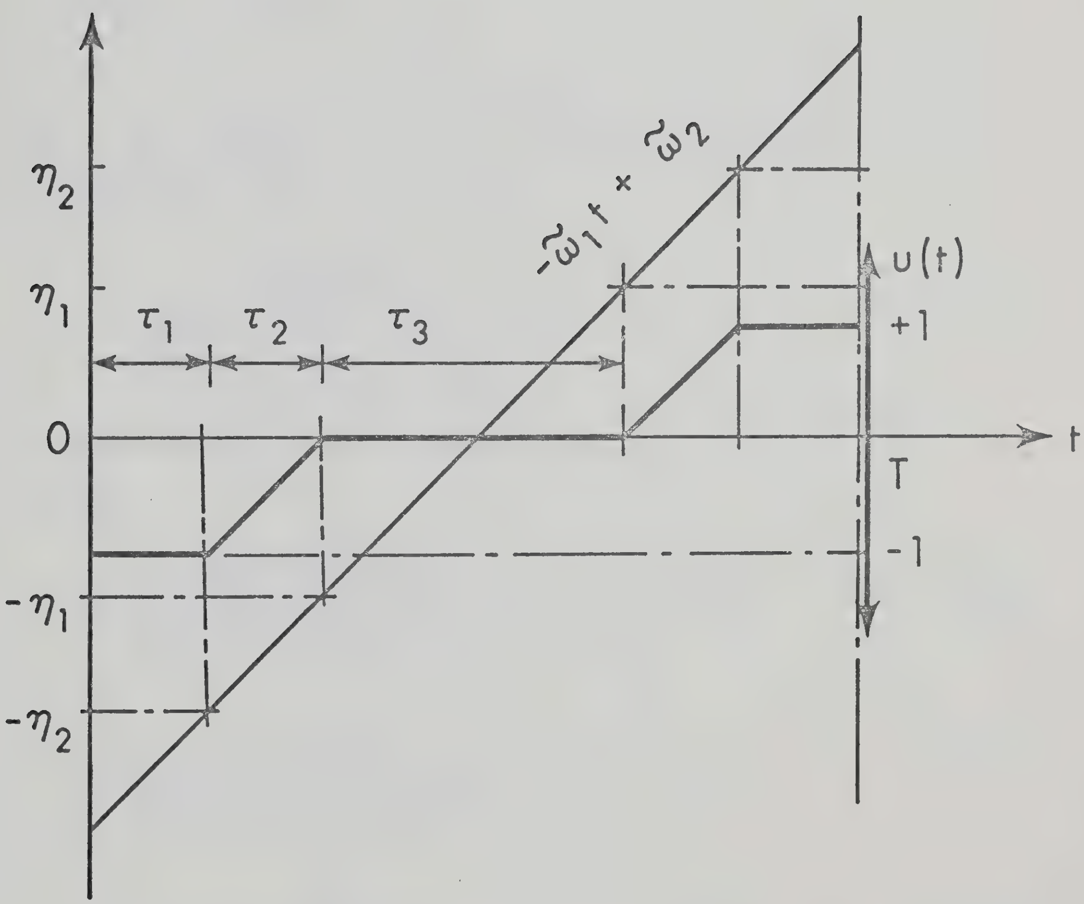


Figure (2.13) Optimal Control for Example (2) (Chapter (2))

$$-\tilde{\omega}_1(\tau_1 + \tau_2) - (1 + c_2/c_1 + c_3/c_1) = -c_2/c_1$$

then,

$$\tau_2/T = (c_3/c_1)/(1 + c_2/c_1 + c_3/c_1) \tag{2.63}$$

Similarly at $t = \tau_1 + \tau_2 + \tau_3$,

$$-\tilde{\omega}_1(\tau_1+\tau_2+\tau_3) - (1 + c_2/c_1 + c_3/c_1) = c_2/c_1,$$

then

$$\tau_3/T = (c_2/c_1)/(1 + c_2/c_1 + c_3/c_1) \tag{2.64}$$

Figure (2.14) shows a typical trajectory passing by $(x_{10},0)$ reaching the origin. By solving along such trajectory we obtain the following relation between the final time and x_{10},

$$T = \frac{1+c_2/c_1+c_3/c_1}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}} \quad 2\sqrt{x_{10}} \tag{2.65}$$

By substituting equation (2.65) into equations (2.62), (2.63), and (2.64) we get

$$\tau_1 = \frac{0.5(1-c_3/c_1)}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}} \quad 2\sqrt{x_{10}} \tag{2.66}$$

$$\tau_2 = \frac{c_3/c_1}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}} \quad 2\sqrt{x_{10}} \tag{2.67}$$

$$\tau_3 = \frac{c_2/c_1}{\sqrt{2(1+c_2/c_1+c_3/c_1-\frac{1}{3}(c_3/c_1)^2-1}} \quad 2\sqrt{x_{10}} \tag{2.68}$$

It is necessary that $0 \leq c_3/c_1 \leq 1$, so that τ_1 is always +ve.
Using equations (2.66), (2.67), and (2.68) we can obtain expressions for $z_1(T) = T$, $z_2(T) = F$, and $z_3(T) = E$,

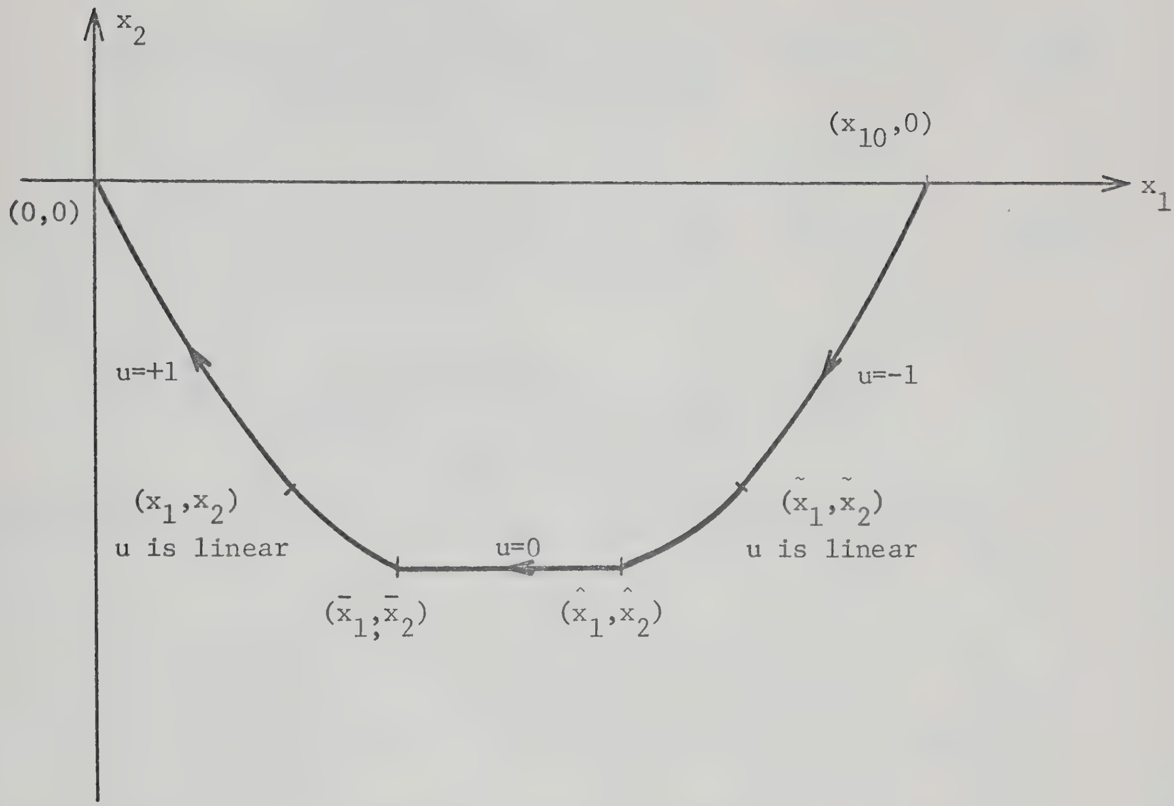


Figure (2.14) A Typical Trajectory for Example (2) (Chapter (2))

$$z_1(T) = \frac{1+c_2/c_1+c_3/c_1}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}}\quad 2\sqrt{x_{10}} \tag{2.69}$$

$$z_2(T) = \frac{1}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}}\quad 2\sqrt{x_{10}} \tag{2.70}$$

$$z_3(T) = \frac{1-(c_3/c_1)/3}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}}\quad 2\sqrt{x_{10}} \tag{2.71}$$

Rewriting expressions τ_1, τ_2, and τ_3 given in (2.66), (2.67) and (2.68) as

$$\left.\begin{aligned} \tau_1 &= \bar{\alpha}\sqrt{x_{10}} \\ \tau_2 &= \bar{\beta}\sqrt{x_{10}} \\ \tau_3 &= \bar{\gamma}\sqrt{x_{10}} \end{aligned}\right\} \tag{2.72}$$

The equations for the switching curves are given by,

$$\begin{aligned} x_1 &= (1/\bar{\alpha}^2 - 0.5)\tilde{x}_2|\tilde{x}_2|, \\ x_1 &= [(1 - \bar{\alpha}^2/2 - \bar{\beta}^2/3 - \bar{\alpha}\bar{\beta})/(\bar{\alpha} + \bar{\beta}/2)^2]\hat{x}_2|\hat{x}_2|, \\ x_1 &= [(1 - \bar{\alpha}^2/2 - \bar{\beta}^2/3 - \bar{\alpha}\bar{\beta} - \bar{\beta}\bar{\gamma}/2 - \bar{\alpha}\bar{\gamma})/(\bar{\alpha} + \bar{\beta}/2)^2]\bar{x}_2|\bar{x}_2|, \end{aligned} \tag{2.73}$$

and

$$x_1 = 0.5x_2|x_2|$$

see figure (2.14).

The general expressions of T, F, and E, for initial states $(x_{10}, x_{20})\varepsilon G^-$ and $(x_{10}, x_{20})\varepsilon G^+$, are given by,

$$T = \pm x_{20} + \frac{1+c_2/c_1+c_3/c_1}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}}\sqrt{4x_{10}+2x_{20}^2}$$

$$F = \pm x_{20} + \frac{1}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}}\sqrt{4x_{10}+2x_{20}^2} \tag{2.74}$$

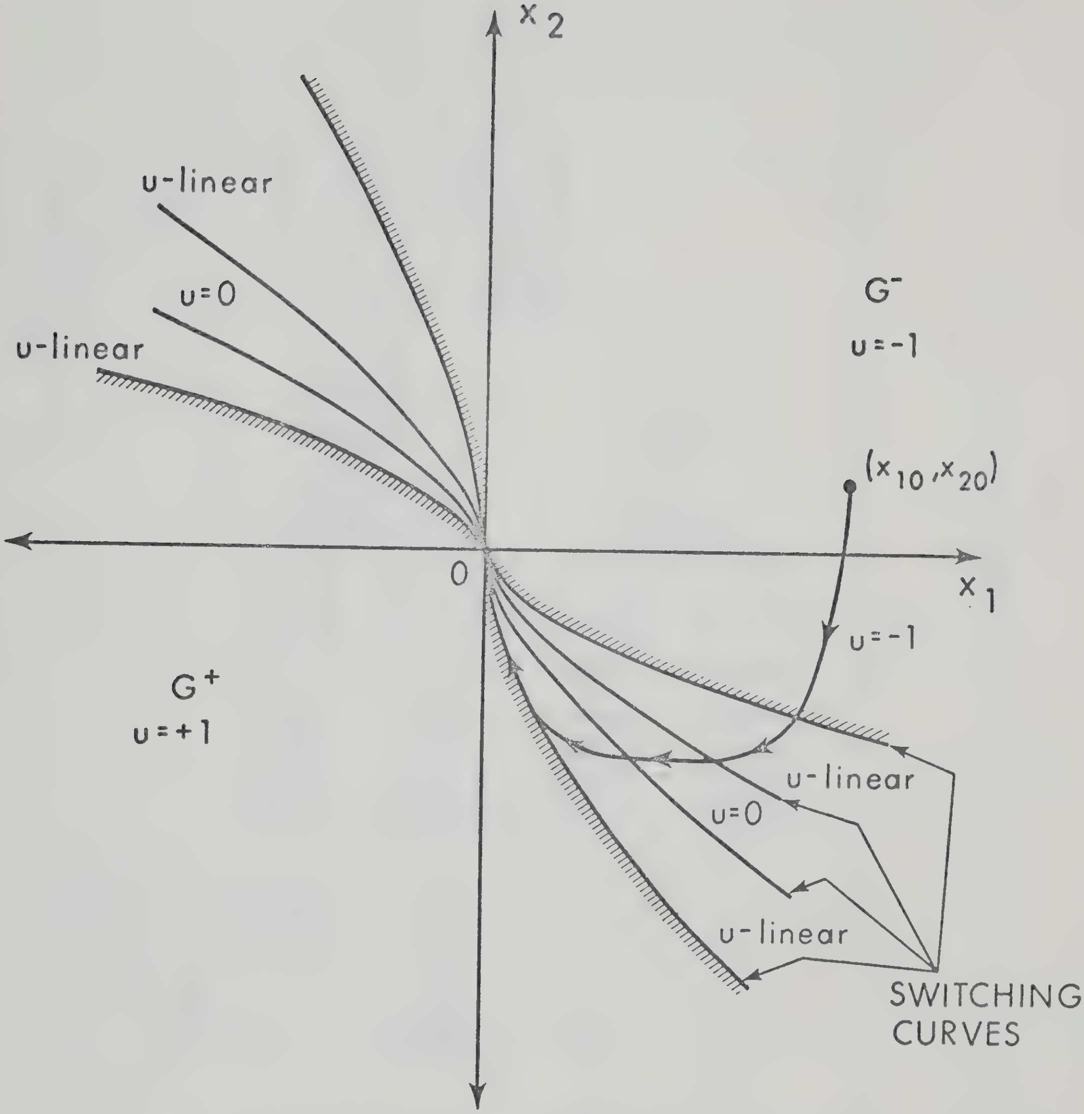


Figure (2.15) Switching Curves for Example (2) (Chapter (2))

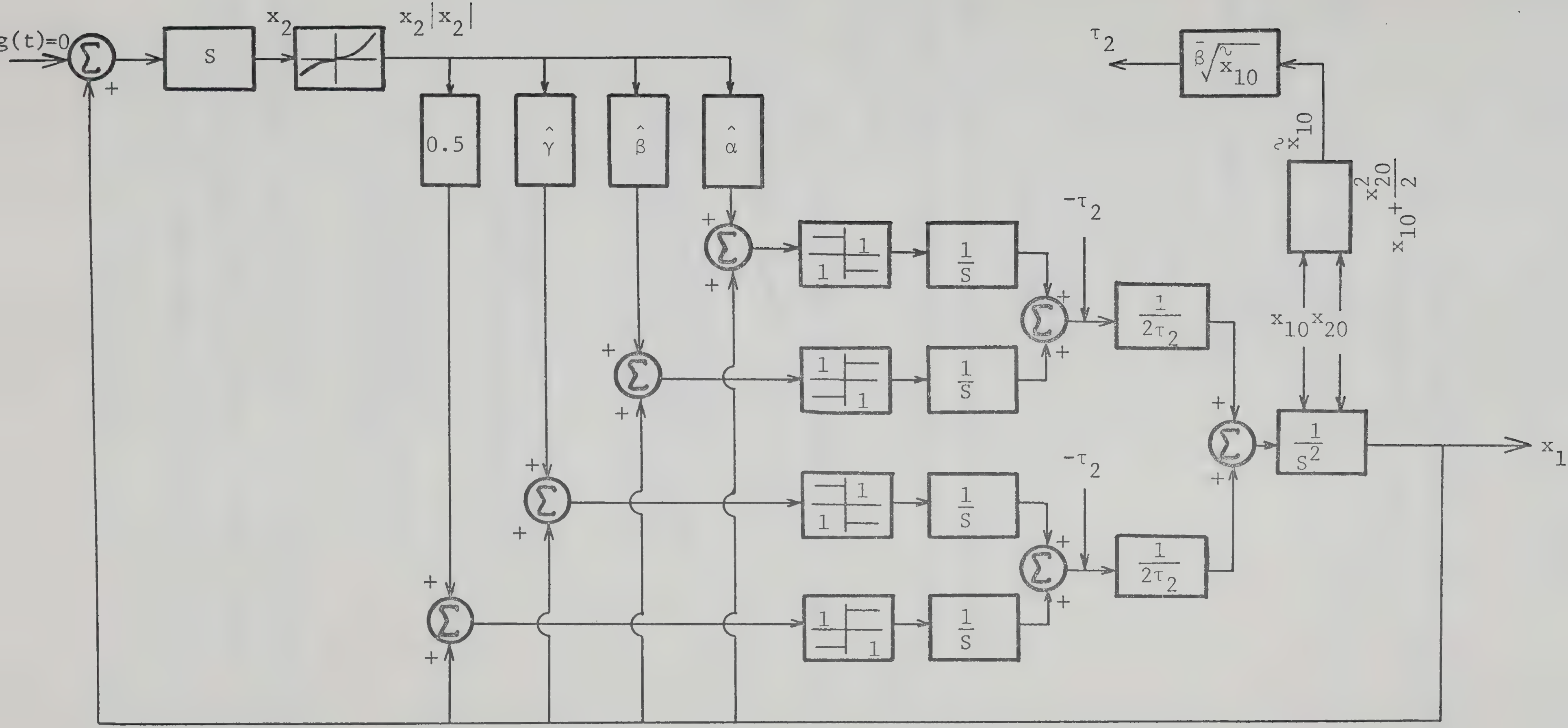


Figure (2.16) Realization of Control Law of Example (2) (Chapter (2))

$$E = \pm x_{20} + \frac{1-(c_3/c_1)/3}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}}\sqrt{4x_{10}+2x_{20}^2}$$

where +ve sign is taken for $(x_{10},x_{20})\varepsilon G^-$ and -ve sign is taken for $(x_{10},x_{20})\varepsilon G^+$, see figure (2.15).

Rewriting equation (2.73) as,

$$\left.\begin{aligned} \tilde{x}_1 &= \hat{\alpha}\tilde{x}_2\ |\tilde{x}_2| \\ \hat{x}_1 &= \hat{\beta}\hat{x}_2\ |\hat{x}_2| \\ \bar{x}_1 &= \hat{\gamma}\bar{x}_2\ |\bar{x}_2|, \text{ and} \\ x_1 &= 0.5x_2\ |x_2| \end{aligned}\right\} \quad (2.75)$$

The realization of the control given in (2.61) is shown in figure (2.16). This is valid only for the initial states $(x_{10},x_{20})\varepsilon G^-$ and $(x_{10},x_{20})\varepsilon G^+$.

As a typical initial condition, let $x_{10}=1$, and $x_{20}=0$. For this initial state, equation (2.74) becomes,

$$\left.\begin{aligned} T &= \frac{2(1+c_2/c_1+c_3/c_1)}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}} \\ F &= \frac{2}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}} \\ E &= \frac{2(1-(c_3/c_1)/3)}{\sqrt{2(1+c_2/c_1+c_3/c_1)-\frac{1}{3}(c_3/c_1)^2-1}} \end{aligned}\right\} \quad (2.76)$$

For the given initial state (1,0), the relation between $z_1(T)$, $z_2(T)$, and $z_3(T)$ as c_2/c_1, and c_3/c_1 vary, is shown in figure (2.17).

It is interesting to note if $c_2/c_1 = c_3/c_1 = 0$ we get the

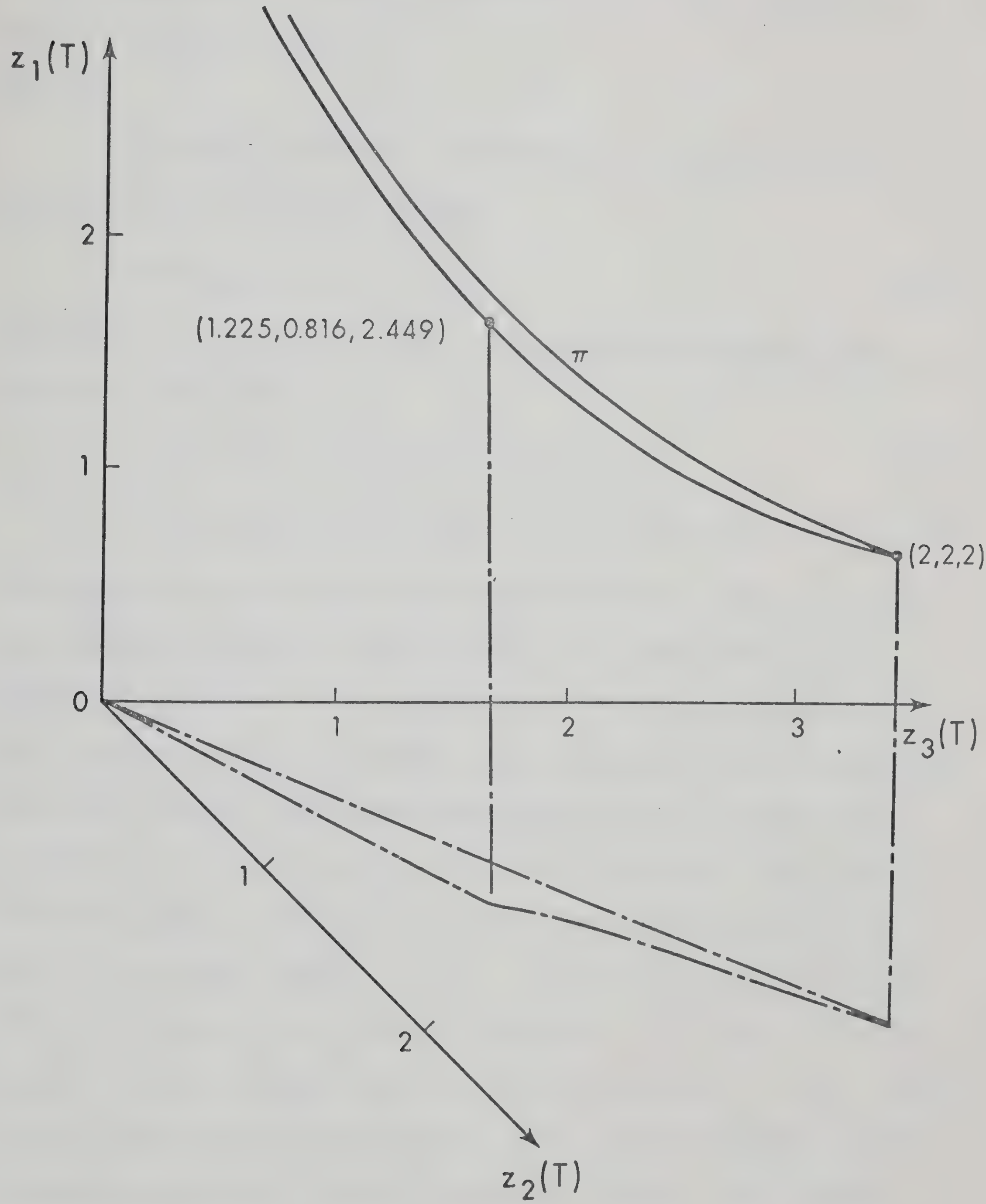


Figure (2.17) Relationship Between $z_1(T)$, $z_2(T)$, and $z_3(T)$ in Example (2) (Chapter (2))

minimum-time problem and if $c_2/c_1 = 0$, we have the case of linear combination of time and energy, and if $c_3/c_1 = 0$, we have the case linear combination of time and fuel which is the case considered in example (1) of this section.

Step (2)

We shall now proceed to determine the optimum values of c_2/c_1 and c_3/c_1, which will minimize

$$\phi\ (z_1,z_2,z_3) = z_1^2+z_2^2+z_3^2 \tag{2.77}$$

The three costs z_1, z_2, and z_3 are directly related because from equation (2.74) we get,

$$c_3/c_1 = 3[1 - (z_3-x_{20})/(z_2-x_{20})], \text{ and} \tag{2.78}$$

$$c_2/c_1 = 3(z_3-x_{20})/(z_2-x_{20}) + (z_1-x_{20})/(z_2-x_{20})-4 \tag{2.79}$$

We can use the Z-space search technique described in section (2.3). z_1 is taken as dependent variable and z_2 and z_3 are taken to be independent variables. For specific values of the coordinates z_2 and z_3 we can get the corresponding c_2/c_1 and c_3/c_1 by using equation (2.78) and (2.76b). The "Accelerating Step Search" technique will be carried out for $x_{10}=1$, $x_{20}=0$. The search consists of two stages. In the 1st stage, we define increments Δz_2 and Δz_3 and evaluate the value of ϕ at $(p\text{x}\Delta z_2,\ q\text{x}\Delta z_3)$, $p = 1,\ldots,M_p$, $q = 1,\ldots,M_q$, where $M_p\Delta z_2$ and $M_q\Delta z_3$ are bounds on z_2 and z_3 respectively. We locate the minimum of ϕ with respect to these discrete values using figure (2.17). Starting from this point the "Accelerating Step" search technique is used to reach the minimum of ϕ within reasonable accuracy. The results obtained for this specific ϕ are

$z_1^*(T) = 2.31601516$

$z_2^*(T) = 1.18737496$

$z_3^*(T) = 1.01407607$

$\phi^*(T) = 7.80213579$

$(c_2/c_1)^* = 0.51268013$

$(c_3/c_1)^* = 0.43785380$

$$[\text{grad } \phi] = \begin{bmatrix} 0.00000000 \\ \\ 0.00000001 \end{bmatrix}$$

The sensitivity of ϕ with respect to variation in (c_2/c_1) and (c_3/c_1) is shown in figure (2.18).

Let us consider the following objective function

$$\phi = 10(z_1-2.0)^2 + z_2 + z_3 \tag{2.80}$$

As mentioned in the previous example, there is no need to formulate a new problem. Just evaluate ϕ at the previous discrete values and locate the minimum. This consists merely of algebraic computation. Then using the "Accelerating Step" search technique using the gradient of the new ϕ to get the optimum values of c_2/c_1 and c_3/c_1. The results of this specific ϕ are:

$z_1^* = 2.16897420$

$z_2^* = 1.36258676$

$z_3^* = 1.22818886$

$\phi^* = 2.87629843$

$(c_2/c_1)^* = 0.29590315$

$(c_3/c_1)^* = 0.29590316$

$$[\text{grad } \phi] = \begin{bmatrix} 0.00000002 \\ 0.00000001 \end{bmatrix}$$

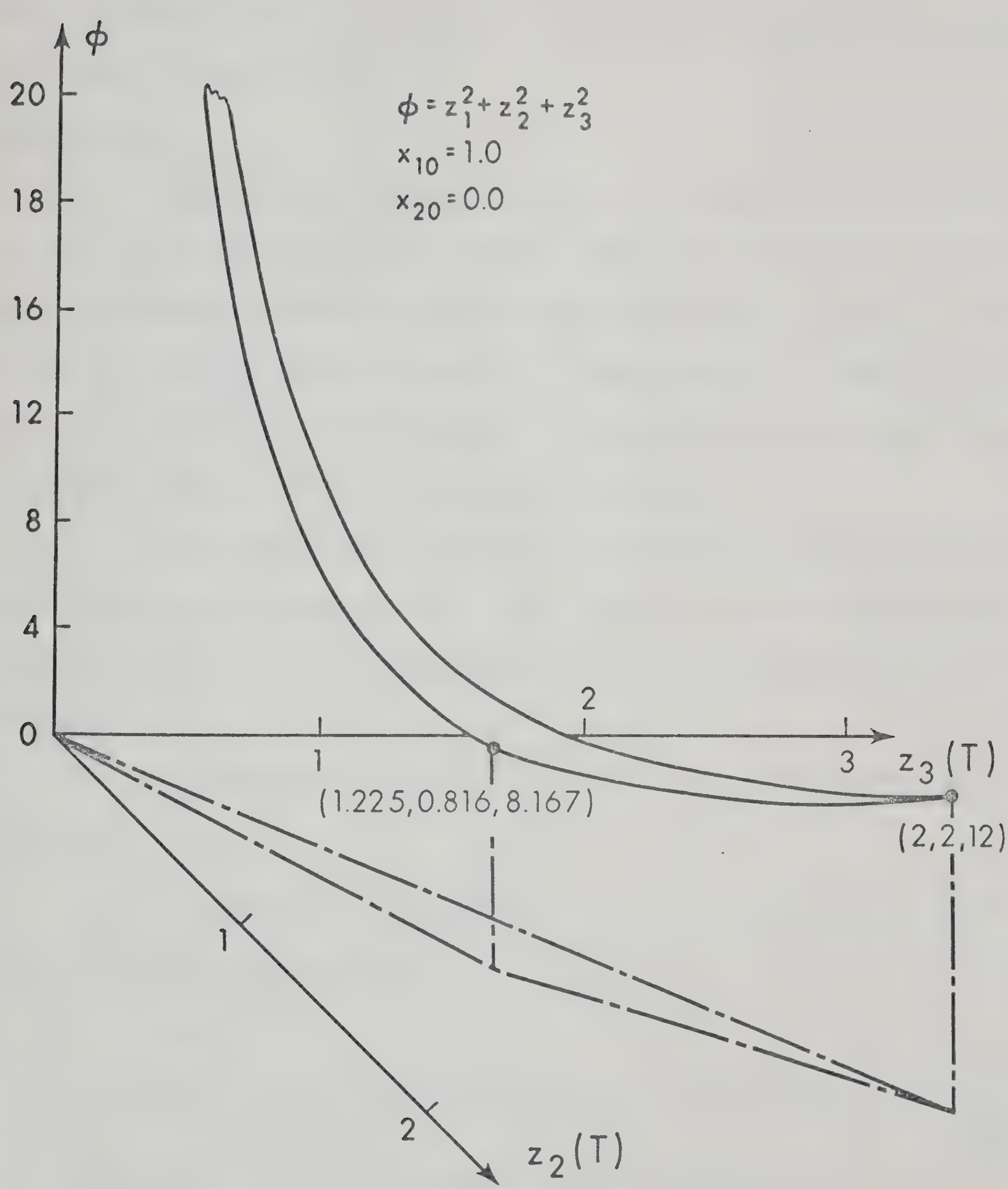


Figure (2.18) Variation of $\phi=z_1^2+z_2^2+z_3^2\Big|_{t=T}$ with Respect to $z_2(T)$ and $z_3(T)$

The sensitivity of ϕ with respect to variations in (c_2/c_1) and (c_3/c_1) is shown in figure (2.19).

Example (3)

The purposes of this example is to illustrate the application of the search technique in the C-Space (the space of weighting factors). The same dynamics and cost scales as given in example (2) will be used. Although the cost scales in example (2) are related (in terms of c_2/c_1 and c_3/c_1) equations (2.78), and (2.79), these relationships will not be needed when the C-Space technique is employed.

The computations carried out in step (1) of example (2) are applicable to this example also. Step (2) is of course different since the search for the minimum value of ϕ will now be carried out in the C-Space.

Following the discussion in section (2.3), for the objective function

$$\phi\ (z_1,z_2,z_3) = z_1^{\ 2}+z_2^{\ 2}+z_3^{\ 2} \tag{2.81}$$

we get, using equation (2.29),

$$[\text{grad}\ \phi] =$$

$$\begin{bmatrix} 2z_1(\partial z_1/\partial(c_2/c_1))+2z_2(\partial z_2/\partial(c_2/c_1))+2z_3(\partial z_3/\partial(c_2/c_1)) \\ 2z_1(\partial z_1/\partial(c_3/c_1))+2z_2(\partial z_2/\partial(c_3/c_1))+2z_3(\partial z_3/\partial(c_3/c_1)) \end{bmatrix} \tag{2.82}$$

Assuming $x_{10} = 1.0$, $x_{20} = 0$. From (2.76) and figure (2.17), the components of the gradient vector are known. Note that the relationships between the cost scales were not used. Using the "Accelerating Step-Search", the results obtained are,

$$z_1{}^* = 2.31601518$$

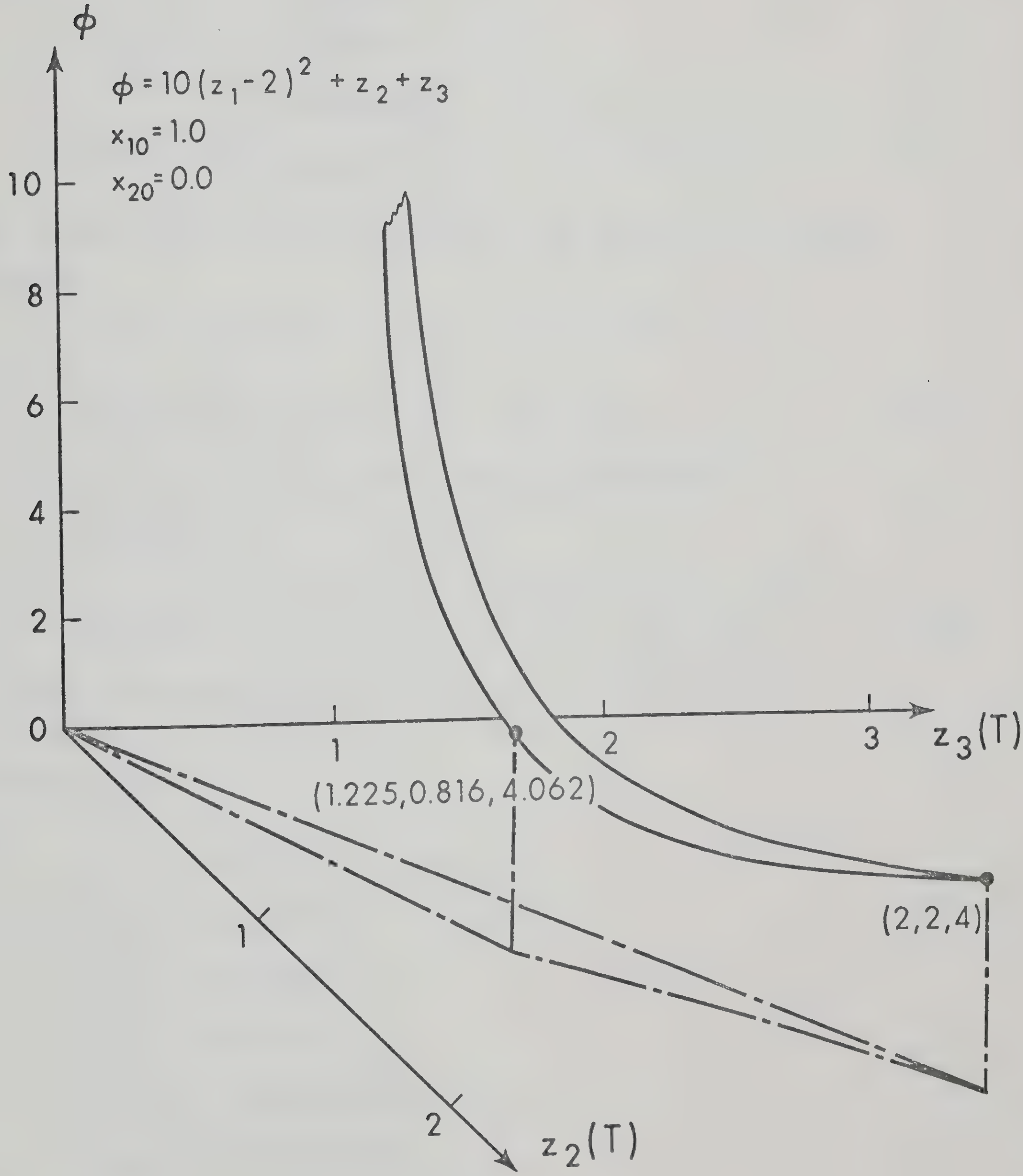


Figure (2.19) Variation of $\phi=10(z_1-2)^2+z_2+z_3\big|_{t=T}$ with respect to $z_2(T)$ and $z_3(T)$

$$z_2^* = 1.18737492$$

$$z_3^* = 1.01407608$$

$$\phi^* = 7.80213578$$

$$(c_2/c_1)^* = 0.51268031$$

$$(c_3/c_1)^* = 0.43785372$$

$$[\text{grad } \phi] = \begin{bmatrix} 0.00000023 \\ 0.00000002 \end{bmatrix}$$

which agree within six decimal places, with the results obtained previously.

For,

$$\phi(z_1,z_2,z_3) = 10(z_1-2.0)^2 + z_2 + z_3 \tag{2.83}$$

and using the same initial conditions, the expression,

$$[\text{grad } \phi] =$$

$$\begin{bmatrix} 20(z_1-2)(\partial z_1/\partial(c_2/c_1)) + \partial z_2/\partial(c_2/c_1) + \partial z_3/\partial(c_2/c_1) \\ 20(z_1-2)(\partial z_1/\partial(c_3/c_1)) + \partial z_2/\partial(c_3/c_1) + \partial z_3/\partial(c_3/c_1) \end{bmatrix} \tag{2.84}$$

yields the results

$$z_1^* = 2.16897416$$

$$z_2^* = 1.36258702$$

$$z_3^* = 1.22818877$$

$$\phi^* = 2.87629843$$

$$(c_2/c_1)^* = 0.29590211$$

$$(c_3/c_1)^* = 0.29590386$$

$$[\text{grad } \phi] = \begin{bmatrix} -0.00000121 \\ -0.00000014 \end{bmatrix}$$

which agree within five decimal places, with the results obtained before.

APPENDIX A-1

Derivation of the SAD Function

Given:

$$f(t).U(t) - a_1.|U(t)| - a_2.U^2(t) \qquad (A1.1)$$

where $U(t)$ is a piecewise continuous function on the interval $[0,T]$, $|U(t)| \leq 1$, and $f(t)$ is defined on the interval $[0,T]$. a_1 and a_2 are +ve constants.

Required:

Find $U(t)$ which maximizes expression (A1.1).

Derivation:

Let us assume $U(t)$ is +ve, then expression (A1.1) becomes,

$$f(t)\ U(t) - a_1.U(t) - a_2.U^2(t) \qquad (A1.2)$$

Differentiating (A1.2) with respect to $U(t)$, we get

$$f(t) - a_1 - 2a_2U(t) \qquad (A1.3)$$

We set (A1.3) equal to zero and solve for $U^*(t)$

$$U^*(t) = (f(t)-a_1)/2a_2 \qquad (A1.4)$$

But $0 \leq U^*(t) \leq 1$. Therefore,

$$\text{if } f(t) < a_1,\ U^*(t) \text{ is zero} \qquad (A1.5)$$

and if $f(t) > a_1+2a_2$, $U^*(t)$ is +1

Similarly for $U(t) < 0$, we get

for $f(t) > -\alpha$, $U^*(t)$ is zero

$$\text{and for } f(t) < -(a_1+2a_2),\ \ U^*(t) \text{ is } -1 \qquad (A1.6)$$

Define the following quantities

$$\eta_1 = a_1$$

$$\eta_2 = a_1 + 2a_2 \qquad (A1.7)$$

Then $U^*(t)$ which maximizes (A1.1) is given by

$$U^*(t) = SAD\ [f(t);\ \eta_1,\eta_2] \qquad (A1.8)$$

$$\mathrm{SAD}\,[f(t);\,\eta_1,\eta_2] = \left[\begin{array}{ll} 1 & f(t)>\eta_2 \\ \dfrac{f(t)-\eta_1}{\eta_2-\eta_1} & \eta_1<f(t)\leq\eta_2 \\ 0 & -\eta_1<f(t)\leq\eta_1 \\ \dfrac{f(t)+\eta_1}{\eta_2-\eta_1} & -\eta_2<f(t)\leq-\eta_1 \\ -1 & f(t)\leq-\eta_2 \end{array}\right\} \quad \text{(A1.9)}$$

This is shown in figure (A1.1).

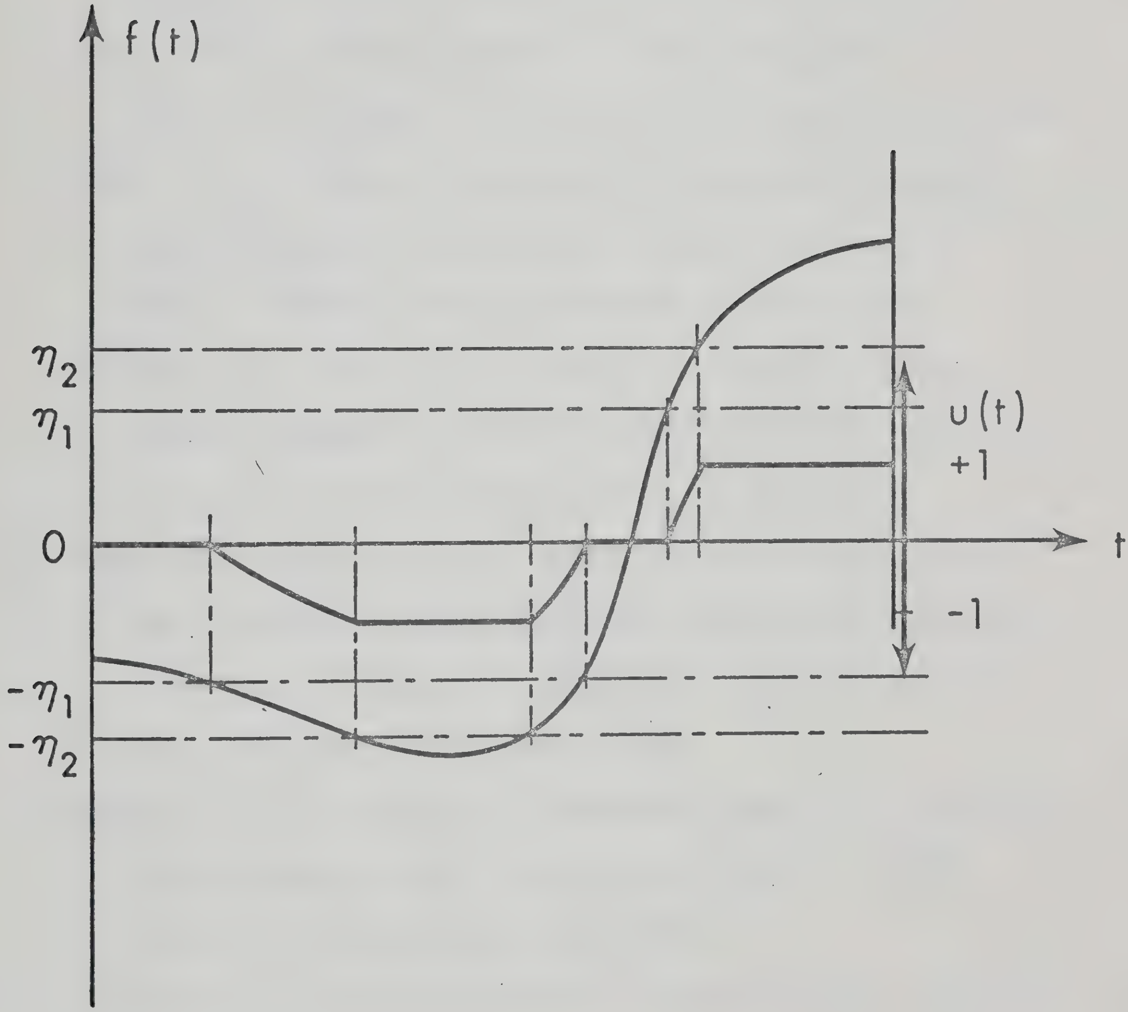


Figure (A-1.1) Saturation and Dead-Zone (SAD) Function

REFERENCES

1. Athans, M. and Falb, P.L., "Optimal Control", New York: McGraw-Hill, 1966.

2. Brooks, S.H., "A Discussion of Random Methods for Seeking Maxima", Operations Research, 6, 244-251, March 1958.

3. Chang , S.S.L., "General Theory of Optimal Processes", SIAM J. on Control, Vol. 4, No. 1, pp. 46-55, February 1966.

4. Gall, D.A., "A Practical Multifactor Optimization Criterion", Recent Advances in Optimization Techniques Edited by Lavi A. and Vogl T.P., pp. 369-386, John Wiley 1966.

5. Kalinin, V.N., "Generalized Optimality Criteria in Optimal Control Problems", Avtomatika i Telemekhanika Vol. 26, No. 2, pp. 365-369, February 1965, (in Russian).

6. Knapp, C.H. and Forst, P.A., "Determination of Optimum Control and Trajectories Using the Maximum Principle in Association with Gradient Technique", IEEE Trans. Automatic Control, Vol. AC-10, pp. 189-193, April 1968.

7. Nelson, W.L., "On the Use of Optimization Theory for Practical Control System Design", IEEE Trans. Automatic Control, Vol. AC-9, pp. 469-478, October 1964.

8. Pierre, D.A., Optimization Theory with Applications , Sec 6.7, John Wiley, 1969.

9. Rozonoer, L.I.," L.S. Pontryagin Maximum Principle in the Theory of Optimum Systems I", Automation and Remote Control, Vol. 20, No. 10, 1959.

10. __________ , "L.S. Pontryagin's Maximum Principle in Optimal System Theory II", Automation and Remote Control, Vol. 20, No. 10, 1959.

11. Waltz, F.M., "An Engineering Approach: Hierarchical Optimization Criteria", IEEE Tans. Automatic Control, Vol. AC-12, pp. 179-180, April 1967.

12. Zadeh, L.A., "Optimality and Non-Scalar Valued Performance Criteria", IEEE Trans. Automatic Control, Vol. AC-8, pp. 59-60, January 1963.

CHAPTER (3)

OPTIMIZATION OF A DETERMINISTIC SYSTEM WITH TWO CONTROL FUNCTIONS AND SEVERAL COST FUNCTIONALS

3.1 Introduction

In this chapter, the objective is to determine optimal controls for a deterministic system with two independent controls, with respect to a performance criterion which in turn can be expressed as a function of several cost scales. The controls will be assumed to be antagonistic in the sense that if one control minimizes the performance criterion the other control would try to maximize it.

Several examples where such a problem would arise can be given. For instance, in the field of economics, the problem of maximizing the productivity of an economy can be formulated along the above lines. The action taken to maximize productivity is treated as one control whereas another factor which adversely affects productivity is taken to be an antagonistic control. In the area of guided missiles, the Pursuit-Evasion problem is another example.

To the best knowledge of the author the previous work in this area appears to be mostly confined to cases where only a single cost functional of the min-max type is used as the performance criterion. This problem has often been formulated as a Zero-Sum Two-Person Differential Game[1,2,3,4,5].

A limitation of the available techniques should be mentioned. If it is required to determine the optimal controls of a system for several objective functions of the min-max type, then the optimal control problem must be solved separately for each objective function.

In some cases, the determination of optimal controls by the available techniques is very difficult [example (2)[2]]. In other cases, the available techniques do not work at all. This point will be illustrated later in the examples.

However, the above difficulties can be overcome by following the procedure described in the previous chapter. It will be shown that there is no need to solve the optimal control problem for each objective function separately. As described in the previous chapter, the two step procedure is used except now, we have to search for a saddle point instead of a min or max since we have a min-max type objective function.

3.2 Mathematical Formulation of the Optimization Problem

Dynamics of the System

The dynamics of the process to be controlled are represented by a set of n-first order ordinary differential equations,

$$\dot{X} = f(X,U,V) \tag{3.1}$$

where ($\dot{X} = dX(t)/dt$). X is an n-dimensional vector representing the state of the system. U and V are the control vectors of the system and are of dimension r_U and r_V respectively. f is an n-dimensional vector valued function, defined over a suitable bounded, connected, open set D of (X,U,V) space, and is assumed to be of class C^2 on its domain of definition. The following information about the system is available to both the controls,

i) the initial value X_o of the state

ii) the dynamics of the system,

iii) the system state, X(t), at time t

The control variables are constrained as follows,

$$\left.\begin{aligned} U(t) &\in K_U \subset E_{r_U} \\ V(t) &\in K_V \subset E_{r_V} \end{aligned}\right\} \quad (3.2)$$

where K_U, K_V are convex and compact, E represents Euclidean space. These control variables U(t) and V(t) are chosen according to a set of rules as follows,

$$\left.\begin{aligned} U(t) &= \hat{U}(X) \\ V(t) &= \hat{V}(X) \end{aligned}\right\} \quad (3.3)$$

where $\hat{U}(X)$ and $\hat{V}(X)$ are members of classes of functions denoted by $\mathcal{U}$ and $\mathcal{V}$ respectively, which are piecewise C^2 and the control variables satisfy (3.2) $\forall t \varepsilon [t_o, T]$, where t_o is the initial time, and T is the final time of the process. The significance of equation (3.3) is that we are interested in obtaining closed loop optimal controls.

A terminal manifold S, which is an n-dimensional, connected manifold of class C^2 in X-space is assumed to be given. The process is to be terminated whenever $X \varepsilon S$. Points on S will be denoted by X_T.

Cost Scales and Objective Function

Let us define the following cost scales,

$$J_k(X,U,V) = g_k(X_T) + \int_{t_o}^{T} l_k(X,U,V)\,d\tau \quad (3.4)$$

$$k = 1,\ldots,N$$

where the functionals g_k's are assumed to be of class C^2 and defined on S and l_k's are of class C^2 and defined on D. Let us define the *"Objective Function"* as $\phi(J_1,\ldots,J_N)$, where ϕ is of class C^2 and defined on S. U is trying to maximize ϕ while V is trying to minimize it.

A pair of controls $(\hat{U},\hat{V}) \varepsilon \mathcal{U} \mathbf{x} \mathcal{V}$ is defined to be a *"Consistent Pair"*, if corresponding to these controls equation (3.1) has a continuous solution defined over some interval $[t_o,T]$, satisfying the initial condition at t_o, and $X_T \varepsilon S$. We will consider only those sub-classes $\mathcal{U}_1 \sqsubset \mathcal{U}$ and $\mathcal{V}_1 \sqsubset \mathcal{V}$ such that for every $\hat{U} \varepsilon \mathcal{U}_1$ and $\hat{V} \varepsilon \mathcal{V}_1$ the pair $(\hat{U},\hat{V})$ is consistent.

Statement of the Optimal Control Problem

The optimal control problem can now be stated as follows:

Find a Consistent pair $(\hat{U}^*,\hat{V}^*)$ *such that*

$$X(T;X_o,\hat{U}^*,\hat{V}^*) \varepsilon S \tag{3.5}$$

and

$$\phi(X,\hat{U},\hat{V}^*) \leq \phi(X,\hat{U}^*,\hat{V}^*) \leq \phi(X,\hat{U}^*,\hat{V}), \tag{3.6}$$

$\forall X, \hat{U} \varepsilon \mathcal{U}_1$ *and* $\hat{V} \varepsilon \mathcal{V}_1$

$(\hat{U}^*,\hat{V}^*)$ is defined to be an *Optimal Pair* of control variables. $X^*(t)$ resulting from $(\hat{U}^*,\hat{V}^*)$ is called *Optimal Trajectory*.

Necessary Conditions for Optimality

Following the discussion of section (2.2) we get,

$$\left.\begin{aligned} \dot{z}_k(t) &= \left(\frac{\partial g_k}{\partial X}\right)' f + l_k \\ z_k(t_o) &= g_k(x_o) \\ k &= 1,\ldots,N \end{aligned}\right\} \tag{3.7}$$

Let us adjoin these variables to the state variables to obtain

$$\begin{bmatrix} \dot{X} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} f \\ \left(\frac{\partial G}{\partial X}\right)' f + L \end{bmatrix}, \quad \begin{bmatrix} X(t_o) \\ Z(t_o) \end{bmatrix} = \begin{bmatrix} X_o \\ G(X_o) \end{bmatrix} \tag{3.8}$$

where Z,G and L are N-dimensional vectors. The objective function $\phi(Z)$ can be rewritten as

$$\phi = \int_{t_o}^{T} \left[\frac{\partial \phi}{\partial z_1}\,[(\frac{\partial g_1}{\partial x})'\ f + l_1] + \ldots + \frac{\partial \phi}{\partial z_N}\,[(\frac{\partial g_N}{\partial x})'\ f + l_N]\right] d\tau$$

$$+ \phi[g_1(X_o),\ldots,g_N(X_o)] \tag{3.9}$$

The second term in the right hand side has no effect on the optimization problem since it is a fixed quantity.

We have now cast the problem under discussion into a format for which the procedure used by Chattopadhyay[2] can be used. The Hamiltonian is given by

$$H = \frac{\partial \phi}{\partial z_1}\,[(\frac{\partial g_1}{\partial X})'\ f + l_1] + \ldots + \frac{\partial \phi}{\partial z_N}\,[(\frac{\partial g_N}{\partial X})'\ f + l_N] + P'f$$

$$+ p_{n+1}[(\frac{\partial g_1}{\partial X})'\ f + l_1] + \ldots + p_{n+N}[(\frac{\partial g_N}{\partial X})'\ f + l_N] \tag{3.10}$$

here P is an n-dimensional vector

and

$$\left.\begin{array}{ll} \dot{P} = -H_X & P(T) = 0 \\ \dot{p}_{n+k} = -H_{z_k} & p_{n+k}(T) = 0 \\ k = 1,\ldots,N & \\ \dot{X} = H_P & X(t_o) = X_o, \end{array}\right\} \tag{3.11}$$

and

$$H_U = H_V = 0, \quad \text{between corners.}$$

Here we allow for the fact that the trajectory $X^*(t)$ may have corners. We assume that $X^*(t)$ is of class C^2 between corners.

From equation (3.11) it is clear that the quantities $(p_{n+k} + \frac{\partial \phi}{\partial z_k})$, $k = 1,\ldots,N$ are constants i.e.

$$\left.\begin{array}{l} p_{n+k} + \left.\frac{\partial \phi}{\partial z_k}\right|_{t=T} = \left.\frac{\partial \phi}{\partial z_k}\right|_{t=T} = c_k \neq 0 \\ k = 1,\ldots,N \end{array}\right\} \tag{3.12}$$

These constants may be positive or negative. Then the expression of H becomes,

$$H = \left[P + \sum_{k=1}^{N} c_k \left(\frac{\partial g_k}{\partial X}\right)\right]' f + \sum_{k=1}^{N} c_k l_k \tag{3.13}$$

Let us define

$$\tilde{P} = P + \sum_{k=1}^{N} c_k \left(\frac{\partial g_k}{\partial X}\right) \tag{3.14}$$

Since $P(T) = 0$, we have

$$\tilde{P}(T) = \sum_{k=1}^{N} c_k \left(\frac{\partial g_k}{\partial X}\right)\Bigg|_{t=T} \tag{3.15}$$

From equation (3.14) and (3.11) we have

$$\dot{\tilde{P}} = -H_X \tag{3.16}$$

where the expression for H becomes

$$H = \tilde{P}'f + \sum_{k=1}^{N} c_k l_k \tag{3.17}$$

Thus we see once again that the problem under discussion with objective function $\phi(z)$ is embedded in the problem with objective function $\sum_{k=1}^{N} c_k z_k$ where the c_k's are parameters to be determined. Summarizing the previous necessary conditions we get,

$$\left.\begin{aligned} &\dot{\tilde{P}}(t) = -H_X, && \tilde{P}(T) = \sum_{k=1}^{N} c_k \left(\frac{\partial g_k}{\partial X}\right)\Bigg|_{t=T} \\ &\dot{X}(t) = H_{\tilde{P}}, && X(t_o) = X_o \end{aligned}\right\} \tag{3.18}$$

where

$$H(X,\tilde{P},\hat{U},\hat{V}) = \tilde{P}'.f + \sum_{k=1}^{N} c_k l_k$$

and

$$H_{\hat{U}} = H_{\hat{V}} = 0, \text{ between corners} \tag{3.19}$$

Equation (3.19) covers the four possibilities $\min_{\hat{U}} \min_{\hat{V}} H$, $\min_{\hat{U}} \max_{\hat{V}} H$, $\max_{\hat{U}} \min_{\hat{V}} H$, and $\max_{\hat{U}} \max_{\hat{V}} H$. In the present discussion we will be interested in $\max_{\hat{U}} \min_{\hat{V}} H$. If there are magnitude constraints on U and V then the necessary conditions given by equation (3.18) and (3.19) become, see [5]

$$\left.\begin{aligned} \dot{\tilde{P}}(t) &= -(H_X + H_{\hat{U}}\hat{U}^*_X + + H_{\hat{V}}\hat{V}^*_X), & \tilde{P}(T) &= \sum_{k=1}^{N} c_k \left(\frac{\partial g_k}{\partial X}\right)\Big|_{t=T} \\ \dot{X} &= H_{\tilde{P}} &, \quad X(t_o) &= X_o, \end{aligned}\right\} \tag{3.20}$$

where

$$H(X,\tilde{P},\hat{U},\hat{V}) = \tilde{P}'f + \sum_{k=1}^{N} c_k l_k$$

and

$$H(X,\tilde{P},\hat{U},\hat{V}^*) \leq H(X,\tilde{P},\hat{U}^*,\hat{V}^*) \leq H(X,\tilde{P},\hat{U}^*,\hat{V}) \tag{3.21}$$

$\forall$ X, $\hat{U}\varepsilon U_1$ and $\hat{V}\varepsilon V_1$

3.3 Example

Let us consider a system with dynamics given by,

$$\dot{x} = ax + b_u u + b_v v \tag{3.22}$$

where

$x(t_o) = x_o$, and a, b_u, $b_v > 0$. The final time T is fixed. There are no constraints on the control variables u and v. It is required to find the optimal controls $u^*(x)$ and $v^*(x)$ such that

$$\min_{\hat{v}^*} \max_{\hat{u}^*} \left[x_T^2 + \left[\int_{t_o}^{T} v^2 d\tau\right]^{1/2} - \left[\int_{t_o}^{T} u^2 d\tau\right]^{1/2} \right] \tag{3.23}$$

The conventional method cannot be applied, since the objective function is not in the form of a single integral. Let us reformulate the problem as follows. Given the system,

$$\dot{x} = ax + b_u u + b_v v$$

and the cost scales

$$\left.\begin{aligned} J_1 &= 0.5\, x_T^2 \\ J_2 &= 0.5 \int_{t_o}^{T} u^2 d\tau, \text{ and} \\ J_3 &= 0.5 \int_{t_o}^{T} v^2 d\tau \end{aligned}\right\} \quad (3.24)$$

Determine the optimal controls u*, v* such that

$$\underset{\hat{v}^*}{\text{Min}}\ \underset{\hat{u}^*}{\text{Max}}\ \left(2J_1 + \sqrt{2\, J_2} - \sqrt{2\, J_3}\,\right) \quad (3.25)$$

<u>Step (1)</u>

Following the discussion in section (3.2), we get

$$\left.\begin{aligned} \dot{x} &= ax + b_u u + b_v v & x(t_o) &= x_o \\ \dot{z}_1 &= x(ax + b_u u + b_v v) & z_1(t_o) &= x_o^2/2 \\ \dot{z}_2 &= 0.5\, u^2 & z_2(t_o) &= 0 \\ \dot{z}_3 &= 0.5\, v^2 & z_3(t_o) &= 0 \end{aligned}\right\} \quad (3.26)$$

The objective function is $c_1 z_1 + c_2 z_2 + c_3 z_3$

The Hamiltonian is given by

$$\begin{aligned} H(x, \tilde{p}, \hat{u}, \hat{v}) \triangleq\ & p(t)(ax + b_u u + b_v v) \\ & + 0.5 c_2 u^2 + 0.5 c_3 v^2 \end{aligned} \quad (3.27)$$

From equation (3.18) we get,

$$\dot{\tilde{p}}(t) = -a\tilde{p}(t) \qquad \tilde{p}(T) = c_1 x(T) \quad (3.28)$$

From equation (3.19) setting

$$\left.\begin{aligned} & H_u = H_v = 0 \text{ we get} \\ & u^* = -(1/c_2)\ b_u \tilde{p}(t) \\ & v^* = -(1/c_3)\ b_v \tilde{p}(t) \end{aligned}\right\} \quad (3.29)$$

where $c_1 > 0$, $c_2 < 0$, $c_3 > 0$

c_2 is -ve to insure the maximization of H with respect to u, c_3 is +ve to insure the H is minimum with respect to v. It is a well known technique to assume

$$\tilde{p}(t) = K(t)x(t) \tag{3.30}$$

Therefore equation (3.29) becomes

$$\left.\begin{aligned}\hat{u}^* &= -(1/c_2)\ b_u K(t)x(t)\\ \hat{v}^* &= -(1/c_3)\ b_v K(t)x(t)\end{aligned}\right\} \tag{3.31}$$

where the scalar K(t) is the solution of the Riccati equation

$$\dot{K}(t) = -2aK(t) + [(1/c_2)b_u^2 + (1/c_3)b_v^2]\ K^2(t) \tag{3.32}$$

with the boundary condition

$$K(T) = c_1 \tag{3.33}$$

Then

$$\int_{K(t)}^{c_1} \{1/[K(t)(A\ K(t) - 2a)]\}\ dK(t) = \int_t^T d\tau \tag{3.34}$$

where $A = (1/c_2)b_u^2 + (1/c_3)b_v^2$

The solution of equation (3.34) becomes

$$K(t) = 2ac_1 \frac{1}{B + (2a-B)\ \mathrm{Exp}[2a(t-T)]} \tag{3.35}$$

where

$$B = (c_1/c_2)b_u^2 + (c_1/c_3)b_v^2$$

or

$$K(t) = c_1 \frac{R}{1 + (R-1)\ \mathrm{Exp}[2a(t-T)]} \tag{3.36}$$

where

$$R = 2a/B$$

Then equation (3.31) becomes,

$$\left.\begin{aligned}\hat{u}^* &= -b_u(c_1/c_2)\,\frac{R}{1+(R-1)\,\mathrm{Exp}[2a(t-T)]}\quad x(t)\\[2ex] \hat{v}^* &= -b_v(c_1/c_3)\,\frac{R}{1+(R-1)\,\mathrm{Exp}[2a(t-T)]}\quad x(t)\end{aligned}\right\}\tag{3.37}$$

By substituting this equation into (3.22) and solving for x(t), we get

$$x(t) = x_o\left[\mathrm{Exp}[-a(t-t_o)]\,\frac{1+(R-1)\,\mathrm{Exp}[2a(t-T)]}{1+(R-1)\,\mathrm{Exp}[2a(t_o-T)]}\right]\tag{3.38}$$

and

$$\left.\begin{aligned}z_1(T) &= 0.5x_o^{\;2}\left[\mathrm{Exp}[-a(T-t_o)]\,\frac{1}{(B/2a)+(1-B/2a)\mathrm{Exp}[2a(t_o-T)]}\right]^2\\[2ex] z_2(T) &= 0.5b_u^2(c_1/c_2)^2x_o^2\left[\frac{1}{(B/2a)+(1-B/2a)\mathrm{Exp}[2a(t_o-T)]}\right]^2\frac{1-\mathrm{Exp}[2a(t_o-T)]}{2a}\\[2ex] z_3(T) &= 0.5b_v^2(c_1/c_3)^2x_o^2\left[\frac{1}{(B/2a)+(1-B/2a)\mathrm{Exp}[2a(t_o-T)]}\right]^2\frac{1-\mathrm{Exp}[2a(t_o-T)]}{2a}\end{aligned}\right\}$$

(3.39)

The realization of the control variables is shown in figure (3.1). The relation between z_1, z_2, and z_3 at t=T for different values of (c_2/c_1) and (c_3/c_1) is shown in figure (3.2). There are some bounds on (c_2/c_1) and (c_3/c_1). These can be obtained by setting u(t) = constant for any value of t, we obtain a relation between (c_3/c_1) and (c_2/c_1).

$$(c_3/c_1) = [\xi + (b_u/b_v)^2](c_2/c_1) - \frac{2a}{b_v^{\;2}\ \mathrm{Exp}[2a(T-t_o) - 1]}\tag{3.40}$$

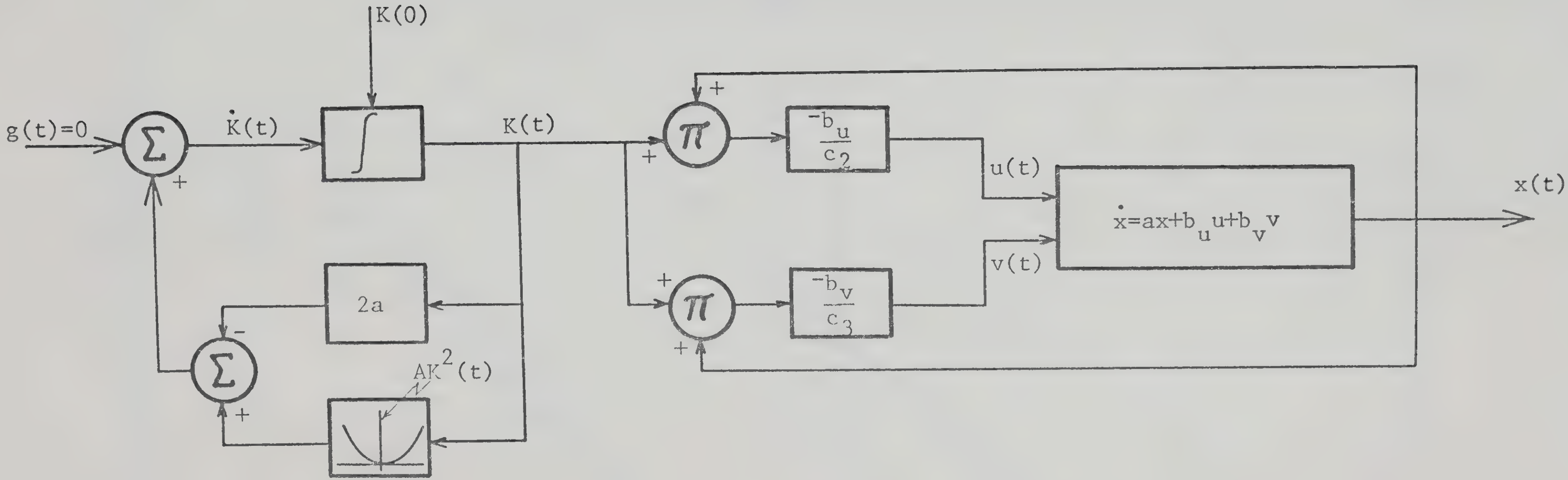
where $0 \leq \xi \leq \infty$

For $\tilde{\xi} = 0$ we get line (1).

For $\tilde{\xi} = \infty$ we get line (2),

see figure (3.3). Similarly from the expression of v(t), we get

$$(c_2/c_1) = [(b_v/b_u)^2 - \tilde{\xi}](c_3/c_1) + \frac{2a}{b_u^{\;2}\mathrm{Exp}[2a(T-t_o) - 1]}\tag{3.41}$$



c_2 is -ve

K(0) is precomputed

Figure (3.1) Realization of Control Functions for the Example in Chapter (3)

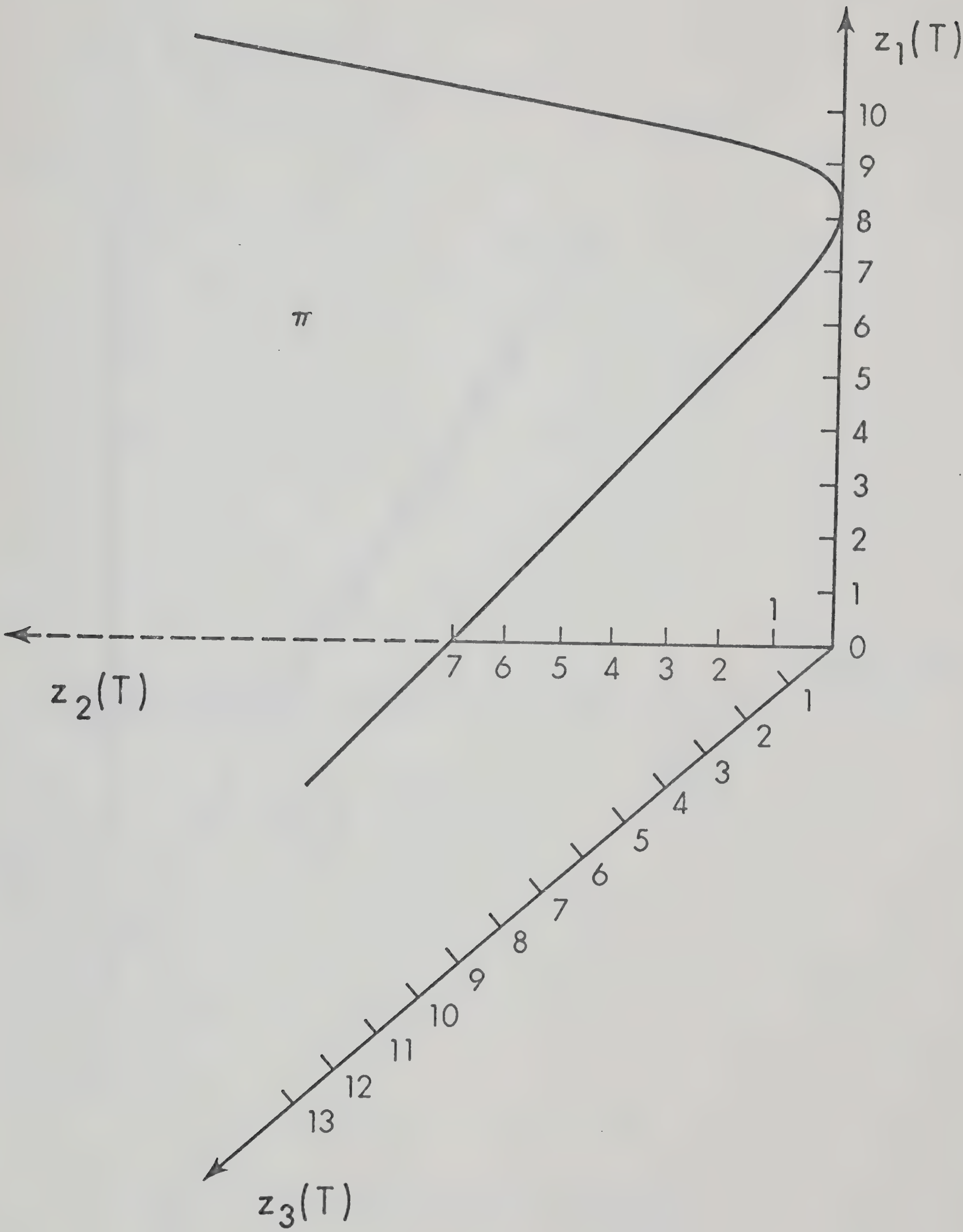


Figure (3.2) Relationship between $z_1(T)$, $z_2(T)$, and $z_3(T)$ in the Example Chapter (3)

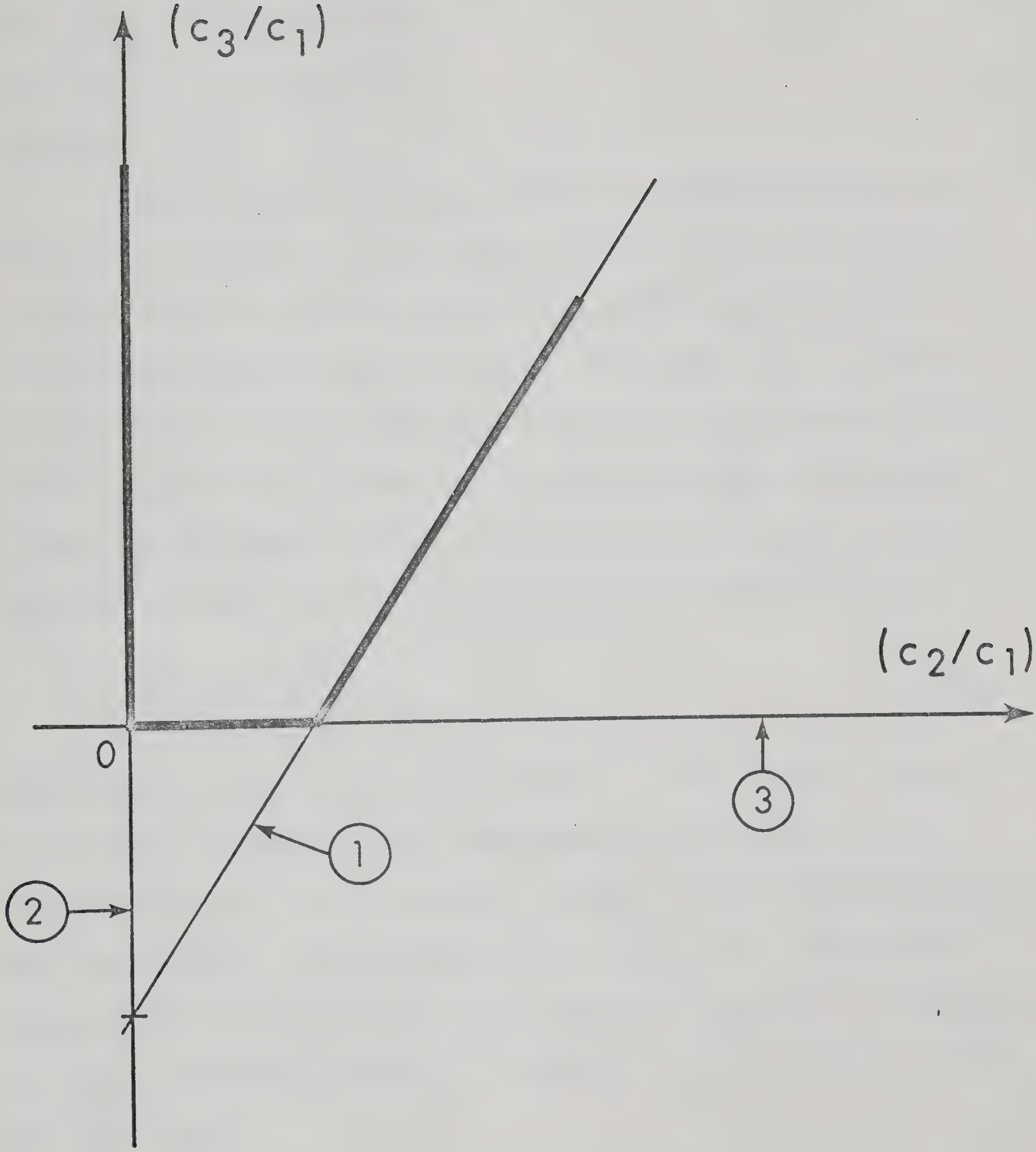


Figure (3.3) Bounds on (c_2/c_1) and (c_3/c_1)

where $0 \leq \tilde{\xi} \leq \infty$

For $\tilde{\xi} = 0$ we get line (1).

For $\tilde{\xi} = \infty$ we get line (3).

Step (2)

Having solved the linear combination problem and determined $\hat{u}^*$ and $\hat{v}^*$ as functions of the ratios c_2/c_1 and c_3/c_1 we now proceed to search for the optimum values of these ratios which correspond to the objective function given by (3.23). This search can be performed in the Z-Space or C-Space depending on whether the cost scales and the ratios c_j/c_1, $j = 2,\ldots,N$ bear any direct relationship to one another or not. In the example under consideration, we shall use z_1 as the dependent variable, z_2 and z_3 as the independent variables. Since

$$\frac{z_2}{z_3} = \frac{b_u^{\,2}(c_2/c_1)}{b_v^{\,2}(c_3/c_1)} \tag{3.42}$$

we can carry out the search in the Z-space. In the previous chapter we considered problems of pure minimization or maximization of the objective function. In this chapter, however, we are considering min-max type problems. Consequently we have to look for a saddle point of the objective function rather than its minimum or maximum. Consequently the search technique described in Chapter (2) will be modified to take this into account.

Applying equation (2.27) to (3.23) we get

$$[\text{grad } \phi] = \begin{bmatrix} \dfrac{\partial\phi}{\partial z_2} \\ \\ \dfrac{\partial\phi}{\partial z_3} \end{bmatrix} = \begin{bmatrix} 2/(c_1/c_2) - 1/\sqrt{2z_2} \\ \\ -2/(c_1/c_3) + 1/\sqrt{2z_3} \end{bmatrix} \tag{3.43}$$

All the components of the gradient vector are known at every point in π, figure (3.2). The procedure for locating the saddle point consists of the following,

1. We specify increments Δz_2 and Δz_3 and compute ϕ at the set of points $\Delta z_2 x p$, $\Delta z_3 x q$, $p = 1,\ldots,M_p$, $q = 1,\ldots,M_q$ where $M_p \Delta z_2$ and $M_q \Delta z_3$ are bounds on z_2 and z_3 respectively. We locate the saddle point of ϕ with respect to these discrete values.
2. Starting from this approximate saddle point, we use a modified form of the "Accelerating Step Method" to get as close as possible to the exact saddle point. The criterion used to determine winning points is the norm of the gradient should be minimum. For the starting point A the winning points are B, C and D. If the starting point is A', the winning points are B', C' and D'. The technique is explained in figure (3.4).

The results obtained for the given objective function,

$$\phi = 2z_1 + \sqrt{2z_3} - \sqrt{2z_2} \tag{3.44}$$

with $a = 0.5$,

$b_u = 0.05$, $b_v = 0.051$,

and $T = 2.0$ are

$(c_1/c_2)^* = 0.0$

$(c_1/c_3)^* = 3.08376312$

$$[\text{grad } \phi] = \begin{bmatrix} 0.00000000 \\ \\ -0.00000000 \end{bmatrix}$$

and the value of the objective function = 16.58586876. All the numerical results reported in this chapter are up to 8 significant figures.

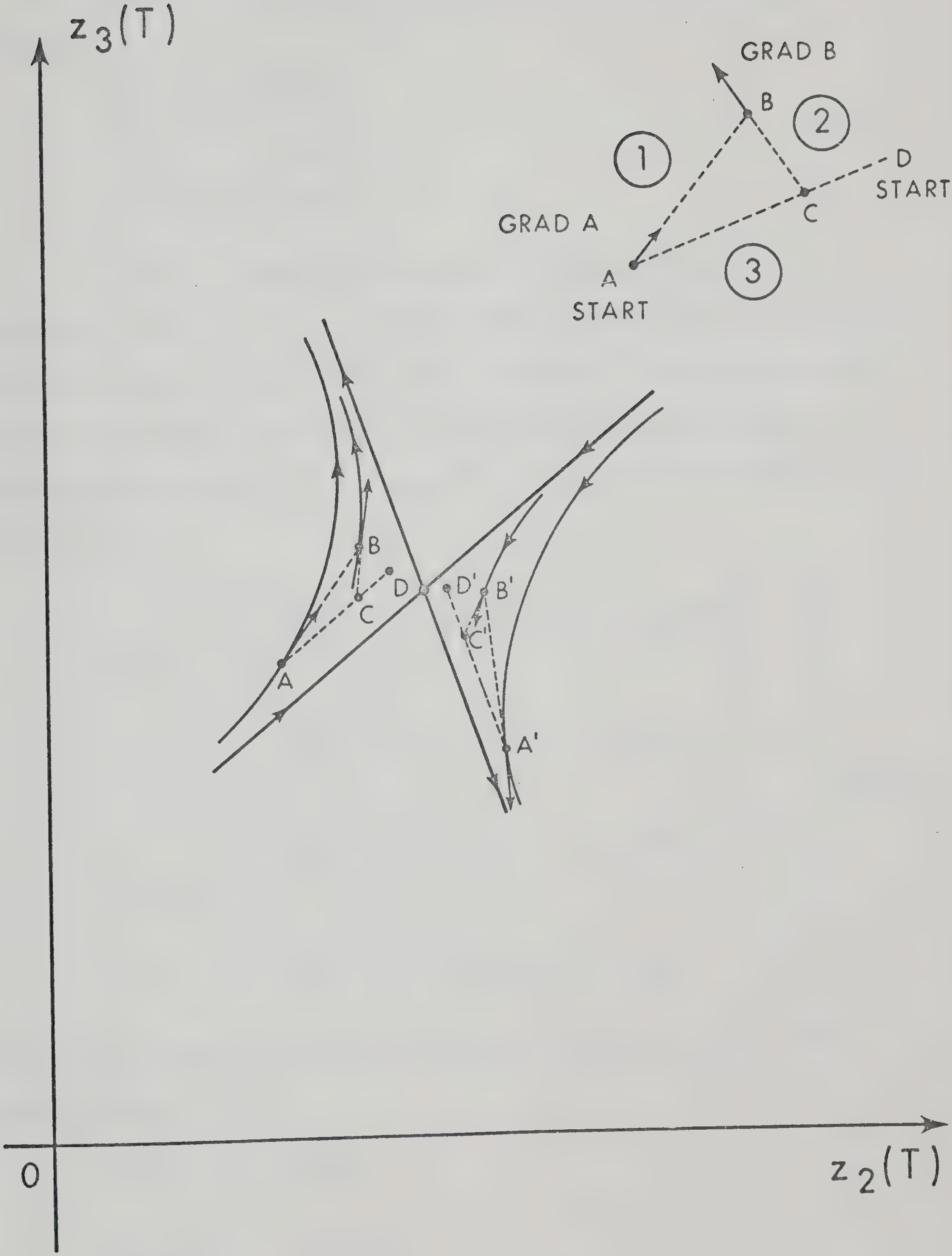


Figure (3.4) Modified Accelerating Step Search

Now let us consider the same system with objective functions given by

$$\left.\begin{aligned} \phi_1 &= z_1 + \frac{z_3}{z_2 + 0.5} \\ \phi_2 &= z_1 + \frac{z_3 + 2.0}{z_2 + 4.0} \end{aligned}\right\} \quad (3.45)$$

There is no need to formulate the new problem from the beginning. Step (1) and part of step (2) (up to the stage of discretization) are common to all objective functions. The approximate value of the saddle point for each of the two new objective functions is determined with respect to the discrete values mentioned earlier. The gradients are,

$$\left.\begin{aligned} [\text{grad } \phi_1] &= \begin{bmatrix} \dfrac{\partial\phi_1}{\partial z_2} \\ \\ \dfrac{\partial\phi_1}{\partial z_3} \end{bmatrix} = \begin{bmatrix} 1/(c_1/c_2) - z_3/(z_2+0.5)^2 \\ \\ -1/(c_1/c_3) + 1/(z_2+0.5) \end{bmatrix} \\ [\text{grad } \phi_2] &= \begin{bmatrix} \dfrac{\partial\phi_2}{\partial z_2} \\ \\ \dfrac{\partial\phi_2}{\partial z_3} \end{bmatrix} = \begin{bmatrix} 1/(c_1/c_2) - (z_3+2)/(z_2+4)^2 \\ \\ -1/(c_1/c_3) + 1/(z_2+4) \end{bmatrix} \end{aligned}\right\} (3.46)$$

The search is then carried out using these gradient vectors. The results obtained are for

$$\phi_1 = z_1 + \frac{z_3}{z_2 + 0.5}$$

$$(c_1/c_2)^* = 7.35984308$$

$$(c_1/c_3)^* = 7.57405060$$

$$[\text{grad } \phi_1] = \begin{bmatrix} -0.00000001 \\ \\ +0.00000003 \end{bmatrix}$$

Value of the objective function = 9.20535621 and for

$$\phi_2 = z_1 + \frac{z_3 + 2}{z_2 + 4}$$

$(c_1/c_2)^* = 6.79863103$

$(c_1/c_3)^* = 9.56304898$

$$[\text{grad } \phi_2] = \begin{bmatrix} 0.00000000 \\ \\ 0.00000000 \end{bmatrix}$$

Value of the objective function = 8.94178680.

REFERENCES

[1] Berkovitz, L.D., "Advances in Game Theory", New Jersey, Princeton University Press, 1964.

[2] Chattopadhyay, R., "On Differential Games", Int. J. Control, Vol. 6, No. 3, pp. 287-295, 1967.

[3] Ho, Y.C., Bryson, A.E., Jr., and Baron, S., "Differential Games and Optimal Pursuit-Evasion Strategies", IEEE Trans. Automatic Control, Vol AC-10, pp. 385-389, October, 1965.

[4] Isaacs, R., "Differential Games", New York: John Wiley, 1966.

[5] Sarma, I.G. and Ragade, R.K., "Some Considerations in Formulating Optimal Control Problems as Differential Games", Int. J. Control, Vol. 4, No. 3, pp. 265-279, 1966.

CHAPTER (4)

OPTIMIZATION OF A TWO-CONTROL MULTI-COST LINEAR SYSTEM WITH JUMP PARAMETERS

4.1 Introduction

In this chapter, the objective is to determine optimal controls for a linear system with two independent controls and several cost functionals and subject to sudden changes in its parameters. The controls will be assumed to be antagonistic in the sense that if one control minimizes the objective function the other control would try to maximize it. The objective function is taken to be the Conditional expectation of a linear combination of several cost functionals with weighting factors unity, positive or negative.

Such problems would arise in situations where there is a possibility of failure of some components of a system or sudden shifts in environment. This problem can be considered to be a stochastic version of a special case of the problem discussed in the previous chapter.

The related work for a single control can be found in [4].

4.2 Mathematical Formulation of the Optimization Problem

Dynamics of the System

The system to be controlled is described by the linear vector differential equation

$$\left.\begin{aligned} \dot{X}(t) &= A(t)\; X(t) + B_U(t)\; U(t) + B_V(t)\; V(t) \\ X(t_o) &= X_o \end{aligned}\right\} \quad (4.1)$$

where ($\dot{X}(t) = dX(t)/dt$)$\cdot X(t)$ is an n-dimensional vector representing the state of the system. $U(t)$ is an r_U-dimensional vector and $V(t)$

is an r_V-dimensional vector are control variables. It is assumed that the number of possible parameter values is finite, say s. We shall represent this variation in parameters by a state Markov jump process. We will assume that each element of the random matrix $[A(t), B_U(t), B_V(t)]$ is a separable Markov process and that there exists an s x s matrix Q such that the conditional probability,

$$P_r\{[A(t+\Delta),B_U(t+\Delta),B_V(t+\Delta)]=[A_j,B_{Uj},B_{Vj}]\,|\,[A(t),B_U(t),B_V(t)]=[A_i,B_{Ui},B_{Vi}]\}$$

$$= \begin{cases} q_{ij}(t)\Delta + o(\Delta) & i \neq j \\ 1 + q_{ii}(t)\Delta + o(\Delta) & i = j, i,j = 1,\ldots,s \end{cases} \qquad (4.2)$$

We denote

$$P_r\{[A(o),B_U(o),B_V(o)]=[A_i,B_{Ui},B_{Vi}]\}=\hat{P}_i, \qquad i = 1,\ldots,s \qquad (4.3)$$

Where $\hat{P}$ is an s-dimensional vector.

For obtaining the optimal controls, the quantity of the data fed back to the controllers must be specified. It will be assumed that the feedback signal will contain both time and the instantaneous state, i.e. (t, X(t)). Furthermore, since the parameters of the plant vary in a random manner, an attempt might be made to monitor these variations. Suppose that there are m sensors on the plant with outputs given by $\mu(t)$. $\mu(t)$ is an m-dimensional random process. This latter process need not be Markovian nor need it be continuous, but it will be assumed to be bounded. It will also be required that $\mu(t)$ is independent of the control variables. Therefore the following information about the process is available to both controllers,

i) the initial value X_o of the state vector

ii) the state X(t) of the system at time t,

iii) the dynamics of the system

iv) the vector $\mu(t)$

It will be assumed that at any time t, the controllers select U(t) and V(t) based on their observation of the vector $(t, X(t), \mu(t))$. In other words

$$\left.\begin{aligned} U(t) &= \hat{U}(t,X(t),\mu(t)) \\ V(t) &= \hat{V}(t,X(t),\mu(t)) \end{aligned}\right\} \quad (4.4)$$

The hat above U and V will be used to emphasize the difference between the function and its value. In other words $\hat{U}$ and $\hat{V}$ represent the controllers and they are therefore non random, while U(t) and V(t) are the output of the controllers and are thus random processes. The controls $\hat{U}$ and $\hat{V}$ are bounded and continuous every where, for $t_o \le t \le T$. The class of admissible control pair is defined as those $\mathcal{U}$: $U(t) = \hat{U}(t,X(t),\mu(t))$ and $\mathcal{V}$: $V(t) = \hat{V}(t,X(t),\mu(t))$. These classes of admissible control pairs are selected to ensure that the system differential equation will be meaningful and integrable when $\hat{U}$ and $\hat{V}$ are inserted in equation (4.1).

Cost Scales

Let us define the following quantities,

$$z_k = \int_{t_o}^{t} l_k\{\tau, X(\tau,t_o,X_o,\hat{U},\hat{V}), \hat{U}(\tau,X(\tau;t_o,X_o,\hat{U},\hat{V}),\mu(\tau)), \hat{V}(\tau,X(\tau;t_o,X_o,\hat{U},\hat{V}),\mu(\tau))\}d\tau \qquad (4.5)$$

where $l_k(t,X,U,V), k = 1,\ldots,N$, are non-negative and continuously differentiable with respect to X,U and V.

Objective Function

Let us define the objective function

$$\phi(\hat{U},\hat{V};t_o,X_o,\mu(t_o)) = E\{\sum_{k=1}^{N} c_k z_k(\hat{U},\hat{V};t_o,X_o,\mu(t_o))_{t=T}\Big| t_o,X_o,\mu(t_o)\} \quad (4.6)$$

where $c_k = +1$ or -1, $k = 1,\ldots,N$, depending on the problem underhand. T is the final time and is fixed. E{ } is the expected value of the quantity between braces. $\hat{U}$ is trying to maximize ϕ and $\hat{V}$ is trying to minimize ϕ.

Statement of the Control Problem

The optimal control problem can now be stated as follows,

Find the optimal pair $(\hat{U}^*,\hat{V}^*)$ *such that*

$$\phi(\hat{U},\hat{V}^*;t_o,X_o,\mu(t_o)) \leq \phi(\hat{U}^*,\hat{V}^*;t_o,X_o,\mu(t_o))$$
$$\leq \phi(\hat{U}^*,\hat{V}\ ;t_o,X_o,\mu(t_o)) \quad (4.7)$$

$\forall\ \hat{U}\varepsilon U$ *and* $\hat{V}\varepsilon V$

The trajectory $X^*(t;t_o,X_o,\hat{U}^*,\hat{V}^*)$ resulting from $(\hat{U}^*,\hat{V}^*)$ is called *"Optimal Trajectory"*.

Necessary Conditions for Optimality

Fix $(t_o,X_o)\varepsilon[o,T] \times E_n$. Define a vector valued random variable $P(t_o,X_o)$ as the solution to the following ordinary differential equation,

$$\frac{d}{d\tau} P(t,X(\tau;t_o,X_o,\hat{U}^*,\hat{V}^*))$$

$$= -\begin{bmatrix} A(\tau) + B_U(\tau)\hat{U}^*_X(\tau,X(\tau;t_o,X_o,\hat{U}^*,\hat{V}^*),\mu(\tau)) \\ + B_V(\tau)\hat{V}^*_X(\tau,X(\tau;t_o,X_o,\hat{U}^*,\hat{V}^*),\mu(\tau)) \end{bmatrix}' .P(\tau,X(\tau;t_o,X_o,\hat{U}^*,\hat{V}^*))$$

$$- \sum_{k=1}^{N} c_k \begin{bmatrix} l_{kX} + l_{kU}\ \hat{U}^*_X(\tau,X(\tau;t_o,X_o,\hat{U}^*,\hat{V}^*),\mu(\tau)) \\ + l_{kV}\ \hat{V}^*_X(\tau,X(\tau;t_o,X_o,\hat{U}^*,\hat{V}^*),\mu(\tau)) \end{bmatrix} \quad (4.8)$$

$$P(T,X) = 0, \forall X,$$

$$t_o \leq \tau \leq T$$

If we define the Hamiltonian to be of the form

$$H(t,X,\hat{U},\hat{V}) = P'(t,X) \cdot \begin{bmatrix} A(t)X + B_U(t)\ \hat{U}(t,X,\mu(t)) \\ + B_V(t)\ \hat{V}(t,X,\mu(t)) \end{bmatrix}$$

$$+ \sum_{k=1}^{N} c_k l_k(t,X,\hat{U}(t,X,\mu(t)),\hat{V}(t,X,\mu(t))) \qquad (4.9)$$

then equation (4.8) can be put in the same form as equation (3.20). It must be emphasized that in the stochastic problem under discussion, the initial value of the Hamiltonian must be specified.

$$H(t_o,X_o,U,V) = P'(t_o,X_o) \cdot \begin{bmatrix} A(t_o)X_o + B_U(t_o)\hat{U}(t_o,X_o,\mu(t_o)) \\ + B_V(t_o)\hat{V}(t_o,X_o,\mu(t_o)) \end{bmatrix}$$

$$+ \sum_{k=1}^{N} c_k l_k(t_o,X_o,\hat{U}(t_o,X_o,\mu(t_o)),\hat{V}(t_o,X_o,\mu(t_o))) \qquad (4.10)$$

where $\hat{U}\varepsilon U$ and $\hat{V}\varepsilon V$.

Based on the results obtained in chapter (3) for the deterministic case and the results obtained by Sworder [4] for a similar problem with a single control, we make the following statement as a necessary condition for optimality.

"We can find optimal controls $\hat{U}^*\varepsilon U$ *and* $\hat{V}^*\varepsilon V$ *such that*

$$E\{H(t,X,\hat{U},\hat{V}^*)\,|\,t,X,\mu(t)\} \leq E\{H(t,X,\hat{U}^*,\hat{V}^*)\,|\,t,X,\mu(t)\}$$

$$\leq E\{H(t,X,\hat{U}^*,\hat{V})\,|\,t,X,\mu(t)\} \qquad (4.11)$$

$\forall \hat{U}\varepsilon U$ *and* $\hat{V}\varepsilon V$."

This means that $\hat{U}^*$ maximizes, and $\hat{V}^*$ minimizes, the conditional expectation of the Hamiltonian. The above statement can be formalized as a theorem. But this will not be attempted here. We proceed to discuss the applications.

4.3 Application

To illustrate the application of the necessary conditions obtained in the previous section. Let us assume the system in (4.1) has three cost scales.

$$\left.\begin{aligned} z_1 &= \frac{1}{2}\int_{t_o}^{t} (X'(\tau)R\,X(\tau))d\tau \\ z_2 &= \frac{1}{2}\int_{t_o}^{t} (U'(\tau)R_U\,U(\tau))d\tau \\ z_3 &= \frac{1}{2}\int_{t_o}^{t} (V'(\tau)R_V\,V(\tau))d\tau \end{aligned}\right\} \quad (4.12)$$

$$t_o \leq \tau \leq T$$

where

R is an n x n positive semidefinite matrix with constant elements,

R_U is an r_U x r_U positive definite symmetric matrix with constant elements,

R_V is an r_V x r_V positive definite symmetric matrix with constant elements.

Let the objective function be

$$\phi(z_1,z_2,z_3) = E\{(z_1+z_3-z_2)_{t=T}\big| t_o,X_o,\mu(t_o)\} \quad (4.13)$$

From here on, we follow Sworder's work for a single control[4] closely and obtain similar results for the case of two controls. The same assumptions made in [4] are made here also. However for sake of being self contained the assumptions will be restated here at the appropriate places.

As a consequence of equation (4.11), see [1], we have

$$\left.\begin{aligned}\hat{U}^*(t,X,\mu(t)) &= E\{R_U^{-1}B_U'(t)\ P(t,X)\,|\,t,X,\mu(t)\}\\ \hat{V}^*(t,X,\mu(t)) &= -E\{R_V^{-1}B_V'(t)\ P(t,X)\,|\,t,X,\mu(t)\}\end{aligned}\right\}\quad(4.14)$$

It is necessary to find an explicit relation for $P(t,X)$ in terms of the vector $(t,X,\mu(t))$. It must be noted that the information available to the controllers is sufficient to determine the instantaneous values of $A(t)$, $B_U(t)$, and $B_V(t)$. Since the sample functions of the elements of $A(t)$, $B_U(t)$, and $B_V(t)$ are not continuous, they are independent of the control law. Therefore, the equations for the optimal controls become,

$$\left.\begin{aligned}\hat{U}^*(t,X,\mu(t)) &= R_U^{-1}\ B_U'(t)\ E\{P(t,X)\,|\,t,X,\mu(t)\}\\ \hat{V}^*(t,X,\mu(t)) &= -R_V^{-1}\ B_V'(t)\ E\{P(t,X)\,|\,t,X,\mu(t)\}\end{aligned}\right\}\quad(4.15)$$

From equation (4.8) we get,

$$\frac{d}{d\tau} P(\tau,X(\tau;t,X,U^*,V^*))$$

$$= -\begin{bmatrix} A(\tau) \\ + B_U(\tau)\hat{U}^*_X(\tau,X(\tau;t,X,\hat{U}^*,\hat{V}^*),\mu(\tau)) \\ + B_V(\tau)\hat{V}^*_X(\tau,X(\tau;t,X,\hat{U}^*,\hat{V}^*),\mu(\tau))\end{bmatrix}' \cdot P'(\tau,X(\tau;t,X,U^*,V^*))$$

$$+ R_U U(\tau)\hat{U}^*_X(\tau,X(\tau;t,X,\hat{U}^*,\hat{V}^*)\mu(\tau))$$

$$- R_V V(\tau)\hat{V}^*_X(\tau,X(\tau;t,X,\hat{U}^*,\hat{V}^*)\mu(\tau))$$

$$- RX \qquad (4.16)$$

where

$$P(T) = 0, \quad t \le \tau \le T$$

To determine the unique solution to (4.16), it is a standard procedure to assume that

$$P(t,X) = K(t)\ X, \qquad 0 \le t \le T \qquad (4.17)$$

where $K(t)$ is a random process independent of X when conditioned on $(t,X,\mu(t))$ and differentiable everywhere.

Substituting equation (4.17) into equation (4.15) we get,

$$\left.\begin{aligned} \hat{U}^*(t,X,\mu(t)) &= R_U^{-1}B_U'(t)E\{K(t)|t,\mu(t)\}X, \\ \hat{V}^*(t,X,\mu(t)) &= -R_V^{-1}B_V'(t)E\{K(t)|t,\mu(t)\}X. \end{aligned}\right\} \quad (4.18)$$

Similarly substituting equation (4.17) into equation (4.16), we get

$$\begin{aligned} &-\left[A'(t)+E\{K'(t)|t,\mu(t)\}B_U(t)R_U^{-1}B'(t)-E\{K'(t)|t,\mu(t)\}B_V(t)R_V^{-1}B_V(t)\right]K(t)X \\ &+B_U'(t)E\{K(t)|t,\mu(t)\}XR_U^{-1}B_U'(t)E\{K(t)|t,\mu(t)\} \\ &-B_V'(t)E\{K(t)|t,\mu(t)\}XR_V^{-1}B_V'(t)E\{K(t)|t,\mu(t)\}-RX \\ &= \dot{K}(t)X + K(t)A(t)X+ K(t)B_U(t)R_U^{-1}B_U'(t)\ E\{K(t)|t,\mu(t)\}X \\ &\qquad - K(t)B_V(t)R_V^{-1}B_V'(t)\ E\{K(t)|t,\mu(t)\}X \end{aligned} \quad (4.19)$$

Since $K(t)$ is conditionally independent of X, and is symmetric with the given boundary conditions,

$$\begin{aligned} \dot{K}(t) &= -A'(t)K(t)-K(t)A(t) \\ &\quad + K(t)[B_V(t)R_V^{-1}B_V'(t)-B_U(t)R_U^{-1}B_U'(t)]\ E\{K(t)|t,\mu(t)\} \\ &\quad + E\{K(t)|t,\mu(t)\}[B_V(t)R_V^{-1}B_V'(t)-B_U(t)R_U^{-1}B_U'(t)]\ K(t) \\ &\quad - E\{K(t)|t,\mu(t)\}[B_V(t)R_V^{-1}B_V'(t)-B_U(t)R_U^{-1}B_U'(t)]E\{K(t)|t,\mu(t)\} \\ &\quad - R \end{aligned} \quad (4.20)$$

where

$$K(T) = 0$$

Since the augmented matrix $[A(t),B_U(t),B_V(t)]$ must be in one of a finite number of different states. The event that $[A(t),B_U(t),B_V(t)]=[A_i,B_{Ui},B_{Vi}]$ will be denoted by $(t,\mu(t))\epsilon[i]$. Define

$$E\{K(t)|(t,\mu(t))\varepsilon[j]\} = \hat{K}_j(t)$$

If at time t, $(t,\mu(t))\varepsilon[j]$, then

$$E\{\dot{K}(t)|(t,\mu(t))\varepsilon[j]\} = -A_j'\hat{K}_j(t) - \hat{K}_j(t)A_j + \hat{K}_j(t)[B_{Vj}R_V^{-1}B_{Vj} - B_{Uj}R_U^{-1}B_{Uj}]\hat{K}_j(t) - R \, . \tag{4.21}$$

From (4.19) it follows that K(t) is bounded on [0,T] and consequently

$$E\{\dot{K}(t)|(t,\mu(t))\varepsilon[j]\} = \lim_{\Delta\to 0} \frac{E\{K(t+\Delta)|(t,\mu(t))\varepsilon[j]\} - E\{K(t)|(t,\mu(t))\varepsilon[j]\}}{\Delta} \tag{4.22}$$

The matrices $A(t),B_U(t),B_V(t)$ are Markovian. Therefore

$$E\{K(t+\Delta)|(t,\mu(t))\varepsilon[j]\} = \sum_{i=1}^{s} \hat{K}_i(t+\Delta)P_r\{(t+\Delta,\mu(t+\Delta))\varepsilon[i]|(t,\mu(t))\varepsilon[j]\} \tag{4.23}$$

By using equation (4.2) we get

$$E\{K(t+\Delta)|(t,\mu(t))\varepsilon[j]\} = \hat{K}_j(t+\Delta)+\Delta \sum_{i=1}^{s} \hat{K}_i(t+\Delta)q_{ji}(t)+0(\Delta) \tag{4.24}$$

From (4.21) and (4.24),

$$\dot{\hat{K}}_j(t) = -A_j'\hat{K}_j(t) - \hat{K}_j(t)A_j + \hat{K}_j(t)[B_{Vj}R_V^{-1}B_{Vj}' - B_{Uj}R_U^{-1}B_{Uj}']\hat{K}_j(t) - \sum_{i=1}^{s} \hat{K}_i(t)q_{ji}(t) - R \tag{4.25}$$

where

$$\hat{K}_j(T) = 0, \; j = 1,\ldots,s$$

The optimal controls are,

$$\left.\begin{aligned}\hat{U}^* &= R_U^{-1} B_{Uj}' \hat{K}_j(t)X \\ \hat{V}^* &= -R_V^{-1} B_{Vj}' \hat{K}_j(t)X\end{aligned}\right\} \quad (4.26)$$

Equations (4.25) and (4.26) are valid

if $(t,\mu(t))\varepsilon[j]$

The solution of the stochastic problem is given by (4.25) and (4.26). The gain matrices $\hat{K}_i(t)$, $i = 1,\ldots,s$, may be obtained by direct integration of equation (4.25). The realization of the optimal controls is shown in figure (4.1). These results are applied to a specific numerical example in the next section.

We will conclude this section by making two observations.

1) As a special case if $A(t)=0, \forall t\varepsilon[t_o,T]$, then we get results for a class of Pursuit-Evasion problem considered by Ho, et al.[2].

2) If the matrices R, R_U, and R_V are also subjected to sudden changes without violating their other properties mentioned before, the technique presented in this section is applicable to include such situations, provided that the vector $\mu(t)$ carries complete information about the instantaneous values of A, B_U, B_V, R, R_U, and R_V.

4.4 A Numerical Example

Consider a system described by the following first order ordinary differential equation,

$$\left.\begin{aligned}\dot{x} &= a(t)\, x + b_u(t)u + b_v(t)v \\ x(t_o) &= x_o\end{aligned}\right\} \quad (4.27)$$

The function a(t) is a Markov jump process with two possible states, 0 and 5, the state 0 is an absorbing state. If a(t) = 5 there is a

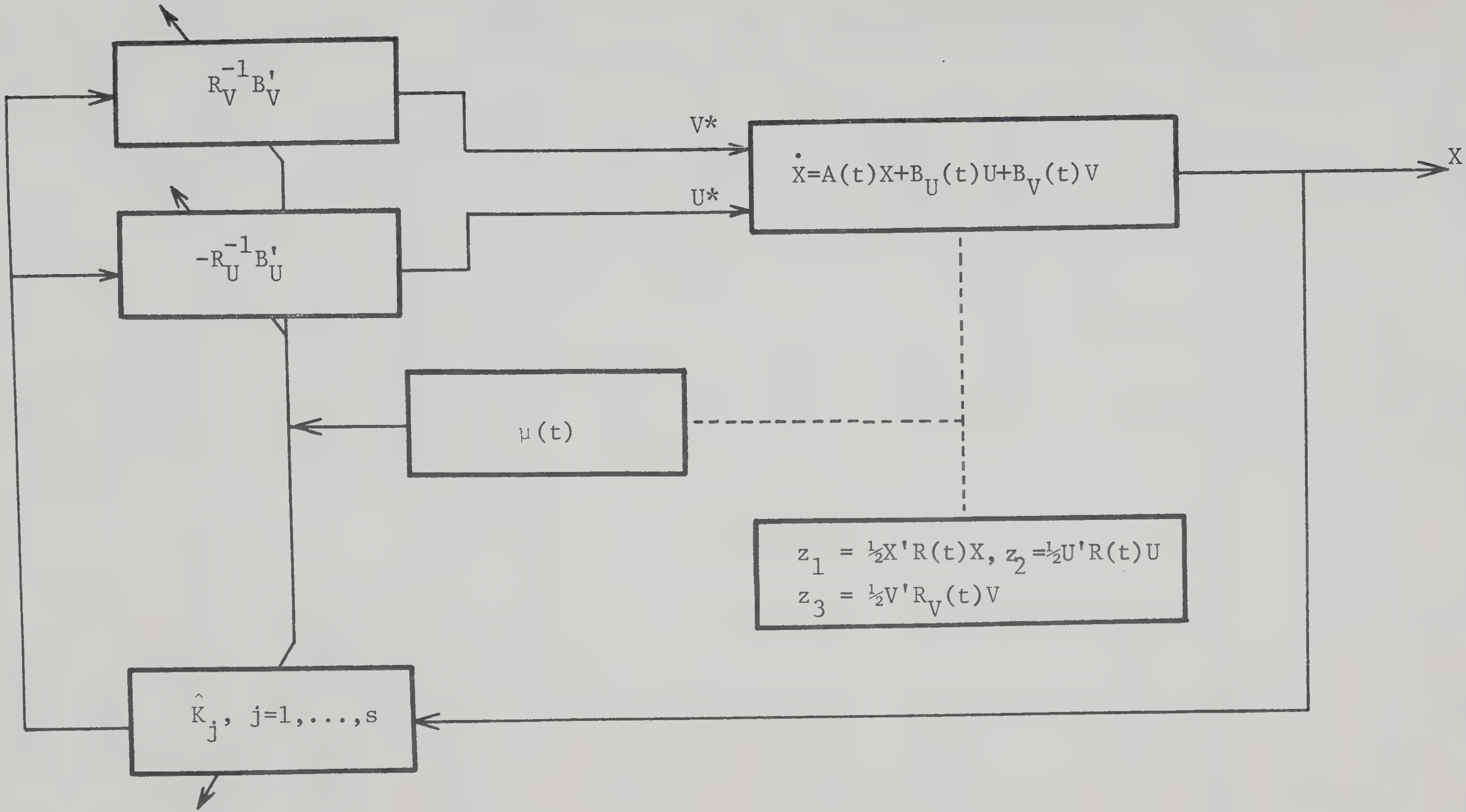


Figure (4.1) Realization of Control Function s for the Example in Chapter (4)

nonzero probability that $a(t+\tau) = 0$, and $\tau>0$. Therefore the Q matrix for $a(t)$ is

$$Q = \begin{bmatrix} 0 & 0 \\ q & -q \end{bmatrix} \qquad q > 0 \tag{4.28}$$

and the initial probability distribution for $a(t)$ is given by,

$$p = \begin{bmatrix} p \\ 1-p \end{bmatrix} \qquad 0 \leq p \leq 1 \tag{4.29}$$

Let the cost scales be given by,

$$\left.\begin{aligned} z_1 &= \frac{1}{2}\int_{t_o}^{\infty} x^2(\tau)d\tau \\ z_2 &= \frac{1}{2}\int_{t_o}^{\infty} u^2(\tau)d\tau \\ z_3 &= \frac{1}{2}\int_{t_o}^{\infty} v^2(\tau)d\tau \end{aligned}\right\} \tag{4.30}$$

and $b_u(t) = 1$ and $b_v(t) = \sqrt{2}$, $\quad \forall t\varepsilon[0,\infty)$

For this specific set of data the controller gains are solutions of,

$$\left.\begin{aligned} \dot{\hat{K}}_1(t) &= +(2-1)\hat{K}_1(t)^2-1, \\ \dot{\hat{K}}_2(t) &= -10\hat{K}_2(t) + (2-1)\hat{K}_2(t)^2 - q\hat{K}_1(t) + q\hat{K}_2(t) - 1 \end{aligned}\right\} \tag{4.31}$$

For any finite value of T, the solution of (4.31) gives the best feedback gains. However, we can make use of the stationary solutions of (4.31) because $T\to\infty$. Thus we get,

$$\left.\begin{aligned} u^*(t,x,\mu(t)) &= \lim_{T\to\infty} 1\ \hat{K}_j(t)x \\ v^*(t,x,\mu(t)) &= \lim_{T\to\infty} \sqrt{2}\ \hat{K}_j(t)x \end{aligned}\right\} \text{if } (t,\mu(t))\varepsilon[j] \tag{4.32}$$

<u>If a(t) = 0</u>

we obtain $\lim\limits_{T\to\infty} K_1(t) = +1$ (4.33)

and

$$\left.\begin{aligned} u^*(t,x,\mu(t)) &= x \\ v^*(t,x,\mu(t)) &= -\sqrt{2}\ x \end{aligned}\right\} \text{if } a(t) = 0 \tag{4.34}$$

and this is the solution of the deterministic problem with $a(t) \equiv 0$

<u>If a(t) = 5</u>

In this case the situation is different. For $a(t) = 5$ we have

$$\lim_{T\to\infty} \hat{K}_2(t) = 5 - \frac{q}{2} + \sqrt{(5 - \frac{q}{2})^2 + 1 + q} \tag{4.35}$$

if $q = 0$,

$$\lim_{T\to\infty} \hat{K}_2(t) = 5 + \sqrt{26}\ ,$$

$$u^*(t,x,\mu(t)) = (5 + \sqrt{26})x \tag{4.36}$$

$$v^*(t,x,\mu(t)) = -\sqrt{2}(5 + \sqrt{26})x$$

and this also is the solution of the deterministic problem with $a(t) = 5$.

If $q = 10$, we get,

$$\left.\begin{aligned} &\lim_{T\to\infty} \hat{K}_2(t) = \sqrt{11} \\ &u^*(t,x,\mu(t)) = \sqrt{11}\ x \\ &v^*(t,x,\mu(t)) = -\sqrt{22}\ x \end{aligned}\right\} \tag{4.37}$$

If $a(t) = 5$, it is clear that for every real number ρ

$$\Pr\left\{ \int_{t_o}^{\infty} \begin{bmatrix} x(\tau;t_o,x_o,u^*,v^*)^2 + v^*(\tau;x(\tau;t_o,x_o,u^*,v^*),u(\tau)^2 \\ - u^*(\tau;x(\tau;t_o,x_o,u^*,v^*),u(\tau)^2 \end{bmatrix} d\tau>\rho \right\} > 0 \tag{4.38}$$

In spite of this

$$\lim_{T\to\infty} x(t;t_o,x_o,u^*,v^*) = 0 \tag{4.39}$$

for all initial conditions. This anomaly is due to the fact that controllability, $\lim_{T\to\infty} x = 0$, of the system when $a(t) = 5$ requires a lot of control "effort". Consequently if one expects the transition in $a(t)$ to occur soon, only the "degree" of uncontrollability is maintained within bounds. For this specific example, this anomaly appears only for cases where $a(t) > 1$.

It was possible to obtain an analytic solution for the problem considered in section (4.3) because we considered a relatively simple objective function. It must be remembered, however, that for a more complex objective function, the search technique described in the previous chapter will have to be used. For instance suppose that the objective is of the form given by,

$$\phi = E\{\phi(z_1,z_2,z_3)_{t=T}\,|\,t_o,x_o,\mu(t_o)\} \tag{4.40}$$

where z_1, z_2, and z_3 are as defined by equation (4.12).

As pointed out in the previous chapter we first solve the so-called linear combination problem (objective function = $E\{(c_1z_1 + c_3z_3 - c_2z_2)_{t=T}\,|\,t_o,X_o,\mu(t_o)\}$ treating the weighting factors c_1, c_2, and c_3 as parameters. The negative sign is used to ensure that we have a min-max type of problem.

The expression of the Hamiltonian is given by,

$$H = P'(t,X) \cdot \begin{bmatrix} A(t)X + B_U(t)\ \hat{U}(t,X,\mu(t)) \\ + B_V(t)\ \hat{V}(t,X,\mu(t)) \end{bmatrix}$$

$$+ \frac{1}{2} c_1\ X'RX + \frac{1}{2} c_3 \hat{V}'R_V\hat{V} - \frac{1}{2} c_2\hat{U}'R_U\hat{U} \tag{4.41}$$

The matrix Ricatti equations given in equation (4.20), now take the form

$$\dot{\hat{K}}_j(t) = -A_j'\hat{K}_j(t) - \hat{K}_j(t)A_j$$

$$+ \hat{K}_j(t)[\frac{1}{c_{j3}} B_{vj}R_V^{-1}B_{Vj}' - \frac{1}{c_{j2}} B_{Uj}R_U^{-1}B_{Uj}']\hat{K}_j(t)$$

$$- \sum_{i=1}^{s} \hat{K}_i(t)q_{ji}(t)$$

$$- c_{j1}R \tag{4.42}$$

where

$$\hat{K}_j(T) = 0, \quad j = 1,\ldots,s$$

where c_{j1}, c_{j2}, and c_{j3} are positive constants.

The optimal controls are

$$\left.\begin{aligned} \hat{U}^* &= \frac{1}{c_{j2}}\ R_U^{-1}B_{Uj}'\hat{K}_j(t)X \\ \hat{V}^* &= -\frac{1}{c_{j3}}\ R_V^{-1}B_{Vj}'\hat{K}_j(t)X \end{aligned}\right\} \tag{4.43}$$

if $(t,\mu(t))\varepsilon[j]$

Let us denote the set of constants (c_{j1},c_{j2},c_{j3}) by C_j. First we determine the numerical values of C_j, $j = 1,\ldots,s$, with respect to the given objective function by treating the problem as a deterministic

problem and using the search technique described in the previous chapter. By substituting these numerical values in (4.42) and (4.43) then equation (4.42) and equation (4.43) give the solution of the stochastic problem. The gain matrices $\hat{K}_j(t)$, $j = 1,\ldots,s$, may be obtained by direct integration of equation (4.42).

REFERENCES

[1] Chattopadhyay, R., "On Differential Games", Int. J. Control, Vol. 6, No. 3, pp. 287-295, 1967.

[2] Ho, Y.C., Bryson, A.E., Jr., and Baron, S., "Differential Games and Optimal Pursuit-Evasion Strategies", IEEE Trans. Automatic Control, Vol. AC-10, pp. 385-389, October 1965.

[3] Sarma, I.G. and Ragade, R.K., "Some Considerations in Formulating Optimal Control Problems as Differential Games", Int. J. Control, Vol. 4, No. 3, pp. 265-279, 1966.

[4] Sworder, D.D., "Feedback Control of a Class of Linear Systems with Jump Parameters", IEEE Trans. Automatic Control, Vol. AC-14, pp. 9-14, Feb. 1969.

CHAPTER (5)

OPTIMIZATION OF A DETERMINISTIC SYSTEM WITH SEVERAL CONTROL FUNCTIONS AND SEVERAL COST FUNCTIONALS

5.1 Introduction

The problem considered in this chapter is concerned with the determination of optimal controls for a deterministic system with several independent controls and several cost scales (say p controls and N cost scales). Each control has several components. In other words each control is a vector. Each cost scale is a function of all controls. The controls will be assumed to be noncooperative in the sense that each control will seek to minimize or maximize its own objective function, each of which in turn is expressed as a function of the cost scales. In this chapter we will assume that each control is maximizing its objective function. Previous work relevant to the discussion here can be found in [2].

Examples where such problems would arise can be found in the field of Economics and Biology. There are some economic situations which can be viewed as non-cooperative-many-player processes [1,2].

5.2 Mathematical Formulation of the Optimization Problem

Dynamics of the System

Let the dynamics of the system to be controlled be represented by the following ordinary vector differential equation,

$$\left.\begin{aligned} \dot{X}(t) &= f(X(t),U_1(t),\ldots,U_p(t)), \\ X(t_o) &= X_o \end{aligned}\right\} \quad (5.1)$$

where ($\dot{X}(t) \equiv dX(t)/dt$). X is an n-dimensional vector representing the state of the system. f is an n-dimensional vector valued function continuous in all its arguments and continuously differentiable with respect to X. U_i, $i = 1,\ldots,p$, are the control variables. U_i has a dimension r_i. The control vectors are required at each instant of time to satisfy certain boundedness constraints of the form,

$$\left.\begin{aligned} \theta_j(U_i) \leq \alpha_j \quad , \qquad & j = 1,\ldots,r_i, \\ & i = 1,\ldots,p \end{aligned}\right\} \qquad (5.2)$$

The control $U_i(t)$ is called an admissible control if,

i) it is piecewise continuous on the control interval $[t_o,T]$, where T is the final time, and

ii) it satisfies (5.2), $\forall t \varepsilon [t_o,T]$.

The set of admissible controls is denoted by $\mathcal{U}_i$, $i = 1,\ldots,p$.

Cost Scales

Let us define the following quantities,

$$z_k = S_k' X \qquad k = 1,\ldots,N \qquad (5.3)$$

which can be rewritten as,

$$\left.\begin{aligned} & \dot{z}_k = S_k' f(X(t), U_1(t),\ldots,U_p(t)), \\ & k = 1,\ldots,N \end{aligned}\right\} \qquad (5.4)$$

where z_k, $k = 1,\ldots,N$ are the cost scales for the system. S_k's are vectors with positive constant elements.

Objective Function

Let us define the objective functions as,

$$\phi_i = \phi_i(z_1,\ldots,z_N), \qquad i = 1,\ldots,p \qquad (5.5)$$

where ϕ_i's are of class C^2.

The control problem can now be stated as follows, find the admissible control functions $U_i^* \varepsilon \mathcal{U}_i$, such that

$$\left.\begin{array}{l}\phi_i(T;X_o,U_1^*,\ldots,U_p^*) \geq \phi_i(T;X_o,\ldots,U_{i-1}^*,\ U_i,\ U_{i+1}^*,\ldots) \\ \forall U_i \varepsilon \mathcal{U}_i,\quad i = 1,\ldots,p\end{array}\right\} \quad (5.6)$$

As indicated earlier, if all the controls, except one, maintain their optimal strategies, then the one deviating from the optimum, may only lower the value of its objective function.

Necessary Conditions for Optimality

Karvovskiy and Kuznetsov[2] obtained necessary conditions for the system given in equation (5.1) for cost scales which can be considered as a special case of equation (5.3). If we let

$$z_k = x_k \qquad k = 1,\ldots,n \tag{5.7}$$

our objective functions will be the same as those treated in [2]. If we augment the cost scales equations given by (5.4) to the system equation (5.1) we can use the necessary conditions in [2] for the more general case considered here.

From equations (5.1) and (5.4) we have,

$$\begin{bmatrix} \dot{X} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} f \\ S'f \end{bmatrix} \quad , \quad \begin{bmatrix} X(t_o) \\ Z(t_o) \end{bmatrix} = \begin{bmatrix} X_o \\ S'X_o \end{bmatrix} \tag{5.8}$$

Where the augmented equation (5.8) is of dimension n+N. By applying the necessary conditions given in [2] we get,

$$H_i(U_1^*,\ldots,U_p^*) = \underset{U_i \varepsilon \mathcal{U}_i}{Max} \quad H_i \tag{5.9}$$

$$\forall t \varepsilon [t_o, T] \qquad i = 1,\ldots,p,$$

where

$$H_i = P_i' f \tag{5.10}$$

and

$$\dot{P}_i(t) = -\frac{\partial H_i}{\partial X} \tag{5.11}$$

$$P_i(T) = \sum_{k=1}^{N} \frac{\partial \phi_i}{\partial z_k} S_k \Big|_{t=T} \tag{5.12}$$

Equation (5.12) can be rewritten as,

$$P_i(T) = \sum_{k=1}^{N} c_{ik} S_k \tag{5.13}$$

where c_{ik} are treated as parameters. Therefore it is clear that the problem with objective functions given by (5.5) is embedded in the linear combination problem. Equation (5.9) through (5.11) and equation (5.13) constitute the necessary conditions for the linear combination problem.

As we have done in the previous chapters, the solution of the problem with objective functions given by (5.5) consists of two steps. The first step is to solve the problem with a linear combination of the cost scales as an objective function, treating the weighting factors as parameters, and the second step is to use the search technique described in Chapter (1) to determine the optimum values of C_i, $i = 1,\ldots,p$.

5.3 Application

Example (1)

Let us consider a system described by the following equations,

$$\left.\begin{aligned} \dot{x}_1 &= ax_1 + u_{11}x_1 \\ \dot{x}_2 &= bx_2 + u_{12}x_1 + u_{22}x_2 \\ x_1(t_o) &= x_{10}, x_2(t_o) = x_{20} \end{aligned}\right\} \tag{5.14}$$

where a and b are positive constants. The control vectors $U_1 = (u_{11}, u_{12})$

and $U_2 = (u_{22})$ are constrained as follows,

$$u_{11} + u_{12} \leq 1 \qquad u_{11} \geq 0 \text{ and } u_{12} \geq 0$$
$$0 \leq u_{22} \leq 1 \tag{5.15}$$

Cost Scales

Let us consider a three cost-scale problem. Let

$$\left.\begin{aligned} z_1 &= x_1 + 2x_2 \\ z_2 &= 2x_1 + x_2 \\ z_3 &= x_2 \end{aligned}\right\} \tag{5.16}$$

Let,

$$\left.\begin{aligned} \phi_1 &= \phi_1(z_1, z_2), \text{ and} \\ \phi_2 &= z_3 \end{aligned}\right\} \tag{5.17}$$

U_1 is maximizing ϕ_1 and U_2 is maximizing ϕ_2.

We will first solve the linear combination problem, by letting

$$\left.\begin{aligned} \hat{\phi}_1 &= c_{11}z_1 + c_{12}z_2, \text{ and} \\ \hat{\phi}_2 &= c_{21}z_3, \quad c_{21} \text{ can be taken} = 1 \end{aligned}\right\} \tag{5.18}$$

For U_2

$$H_2 = p_{21}(ax_1 + u_{11}x_1) + p_{22}(bx_2 + u_{12}x_1 + u_{22}x_2) \tag{5.19}$$

and

$$\left.\begin{aligned} \dot{p}_{21} &= -\frac{\partial H_2}{\partial x_1} = -(a+u_{11})p_{21} - u_{12}p_{22}, \quad p_{21}(T) = 0 \\ \dot{p}_{22} &= -\frac{\partial H_2}{\partial x_2} = -(b+u_{22})p_{22} \qquad p_{22}(T) = 1 \end{aligned}\right\} \tag{5.20}$$

As a consequence of equation (5.9), assuming x_2 is positive, we get

$$U^*_{22}(t) = 1 \tag{5.21}$$

For U_1

$$H_1 = p_{11}(ax_1 + u_{11}x_1) + p_{12}(bx_2 + u_{12}x_1 + u_{22}x_2) \tag{5.22}$$

and

$$\left.\begin{aligned}\dot{p}_{11} &= -\frac{\partial H_1}{\partial x_1} = -(a+u_{11})p_{11} - u_{12}p_{12}, \quad p_{11}(T) = c_{11} + 2c_{12} \\ \dot{p}_{12} &= -\frac{\partial H_1}{\partial x_2} = -(b+u_{22})p_{12} \qquad\qquad p_{12}(T) = 2c_{11} + c_{12}\end{aligned}\right\} \quad (5.23)$$

As a consequence of equation (5.9), assuming x_1 is positive, we get

$$\left.\begin{aligned}&u_{11} = 0 \quad \text{and} \quad u_{12} = 1 \quad \text{if } p_{11} < p_{12} \\ &u_{11} = 1 \quad \text{and} \quad u_{12} = 0 \quad \text{if } p_{11} > p_{12} \\ &u_{11} = \ldots \quad \text{and} \quad u_{22} = \ldots \quad \text{if } p_{11} = p_{12} \text{ singular case}\end{aligned}\right\} \quad (5.24)$$

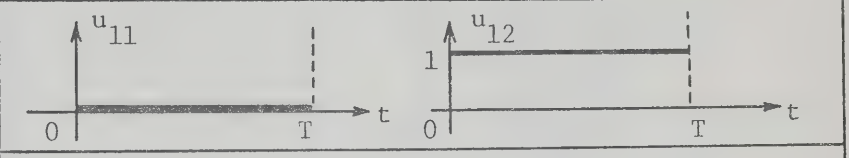
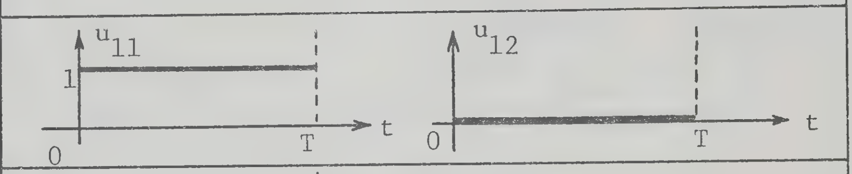
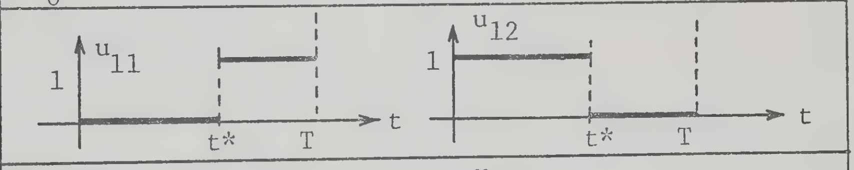
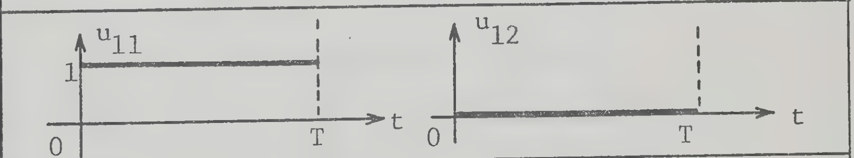
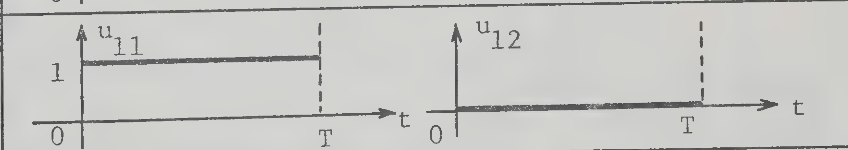
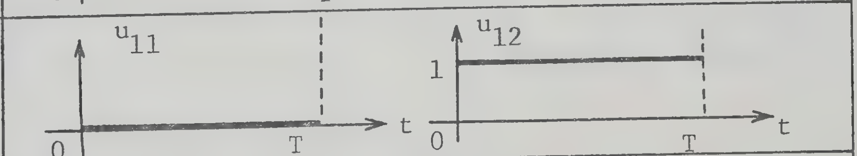
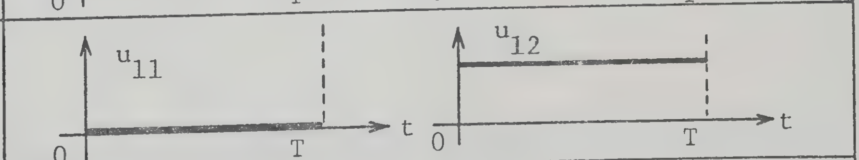
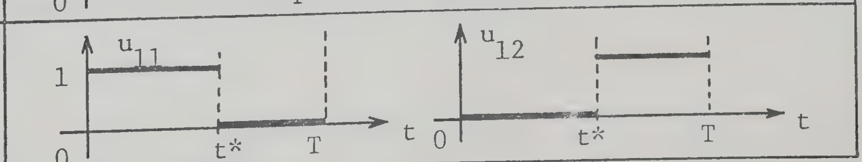
The results obtained for this example are summarized in table (5.1)

Having solved the linear combination problem we can determine the optimum values of c_{11} and c_{12} for any specified objective function ϕ_1 by using the search technique described in Chapter (1). Since this has been demonstrated in earlier chapters it will not be carried out here again.

The preceeding simple example has illustrated to some extent the usefulness of the method developed in this chapter. It must be mentioned that more complex problems can be treated by this method. For instance a multilevel control problem is posed in example (2) and some preliminary results which have been obtained so far are reported.

Example (2) A Multi Level Control Problem

A multi level control system is considered in this example. See figure (5.1). The system has K levels, and level $\ell(\ell = 1,\ldots,K)$ has q_ℓ sectors. The lower levels sectors receive controls from the higher levels but not vice versa. However, there is interaction in both directions between sectors at the same level. Examples of such

$\frac{\alpha}{\beta} = 1$	a<b	
	a=b	
	a>b	
$\frac{\alpha}{\beta} > 1$	a<b	
	a=b	
	a>b	
$\frac{\alpha}{\beta} < 1$	a<b	
	a=b	
	a>b	

$\alpha = c_{11} + 2c_{12}$ $\beta = 2c_{11} + c_{12}$

Table (5.1)

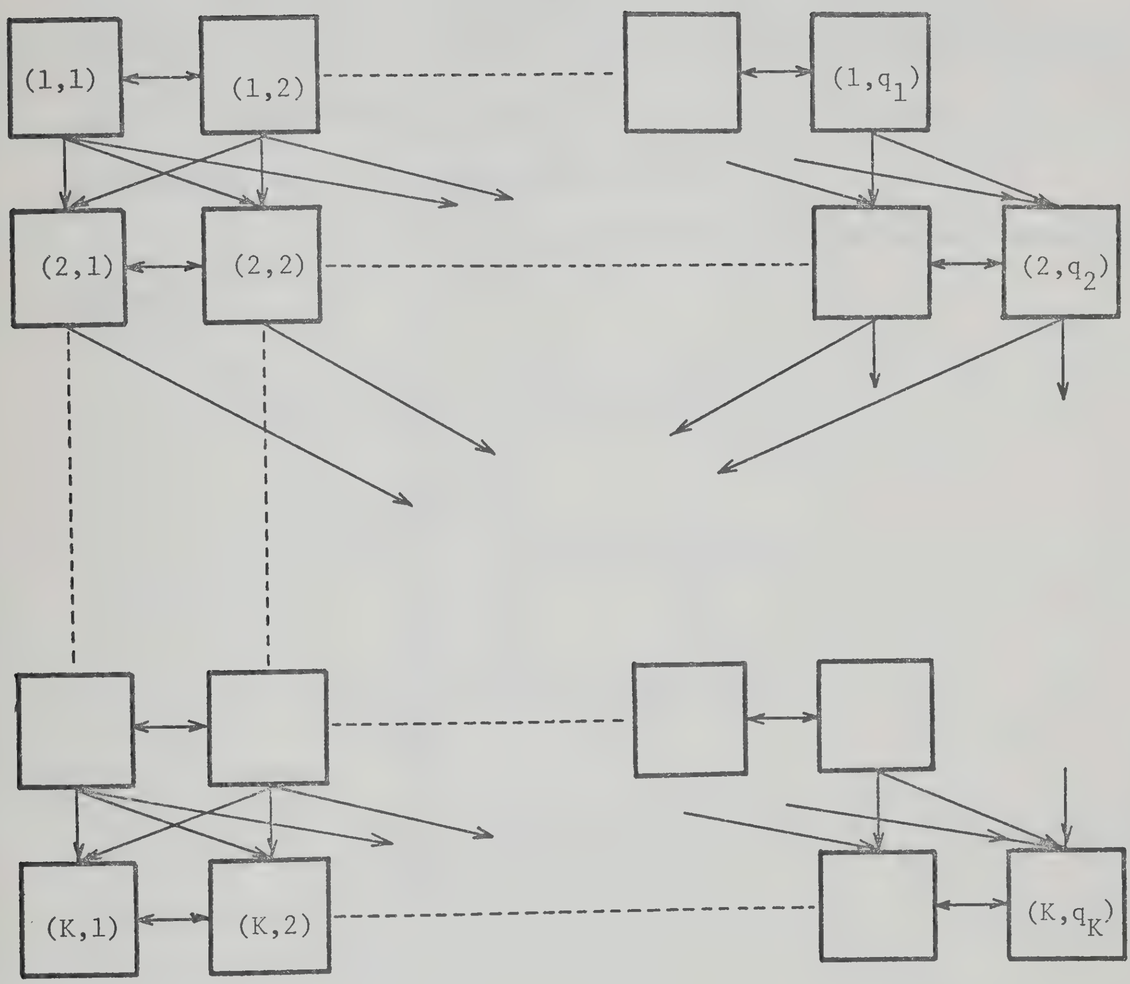


Figure (5.1) A Multi-Level System

systems may be found in the field of Economics. Karvovsky and Kuznetsov[2] consider a single level system with several interacting sectors. Consequently, it will be possible to use their notation and terminology in the present discussion.

Let the dynamics of the system considered be expressed as follows,

$$\dot{X}_{(\ell,m)} = \sum_{s=1}^{q_1} AU_{(1,s)(\ell,m)} X_{(1,s)}$$

$$+ \sum_{s=1}^{q_2} AU_{(2,s)(\ell,m)} X_{(2,s)}$$

$$\vdots$$

$$+ \sum_{s=1}^{q_\ell} AU_{(\ell,s)(\ell,m)} X_{(\ell,s)}$$

$$+ \sum_{s=1}^{q_1} B_{(1,s)(\ell,m)} X_{(1,s)} + \ldots + \sum_{s=1}^{q_K} B_{(K,s)(\ell,m)} X_{(K,s)}$$

$$+ D_{(\ell,m)}$$

$$X_{(\ell,m)}(0) = X_{(\ell,m)_o}$$

$$\ell = 1,\ldots,K, \qquad m = 1,\ldots,q_\ell \tag{5.25}$$

Some explanation of the significance of the symbols used seems to be in order here.

$\dot{X}_{(\ell,m)}(t)$	$dX_{(\ell,m)}(t)/dt$, where
$X_{(\ell,m)}$	is the state vector of sector (ℓ,m), and of dimension $n_{(\ell,m)}$.
$AU_{(1,s)(\ell,m)}X_{(1,s)}$	is a column matrix which represents the contribution of the sector $(1,s)$ to the rate of change of

the state of sector (ℓ,m).

$AU_{(1,s)(\ell,m)}$ is a rectangular matrix with a typical element $a_{(1,s)_\gamma(\ell,m)_{\bar\gamma}}\, u_{(1,s)_\gamma(\ell,m)_{\bar\gamma}}$ where $a_{(1,s)_\gamma(\ell,m)_{\bar\gamma}}$ is a coefficient.

$u_{(1,s)_\gamma(\ell,m)_{\bar\gamma}}$ is a component of the required strategy of sector $(1,s)$.

$u_{(1,s)_\gamma(\ell,m)_{\bar\gamma}} x_{(1,s)_\gamma}$ is a fraction of the state $x_{(1,s)_\gamma}$ contributed to the rate of change of state $x_{(\ell,m)_{\bar\gamma}}$, $\gamma = 1,\ldots,n_{(1,s)}$ and $\bar\gamma = 1,\ldots,n_{(\ell,m)}$

$B_{(1,s)(\ell,m)}$ a rectangular matrix with constant elements, is the coupling matrix between state of sector $(1,s)$ and rate of change of the state of sector (ℓ,m)

$D_{(\ell,m)}$ is a column matrix which represents the free terms.

Figure (5.2) shows schematic representation for special case $K = 2$, $q_1 = 1$, and $q_2 = 2$. It must be noted that the arrow at the input represents the effect of the BX terms.

It is assumed that the strategies of the sectors are constrained as follows,

$$\underbrace{\sum_{k=\ell}^{K}}_{\text{level}} \; \underbrace{\sum_{p=1}^{q_k}}_{\text{sector}} \; \underbrace{\sum_{\rho=1}^{n_{(k,p)}}}_{\text{state variable}} u_{(\ell,m)_{\bar\gamma}(k,p)_\rho} \leq 1 \qquad (5.26)$$

where

$\ell = 1,\ldots,K$

$m = 1,\ldots,q_\ell$

$\bar\gamma = 1,\ldots,n_{(\ell,m)}$

the set of allowable strategies for sector (ℓ,m) is $U_{(\ell,m)}$.

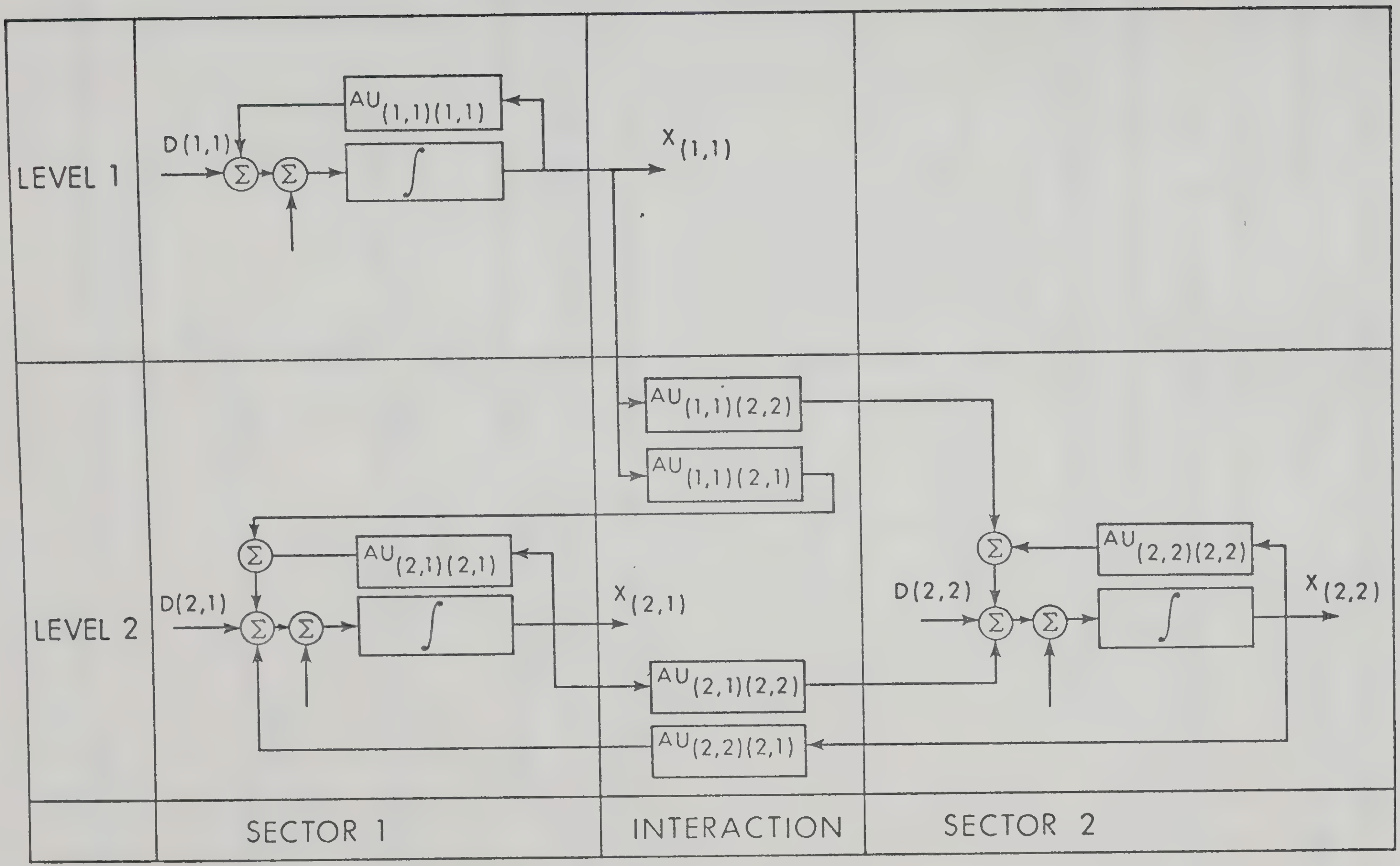


Figure (5.2) Block-Diagram for a Special Case of the Multi-Level System shown in Figure (5.1)

As we have done several times before, we are going to consider first the problem with linear combination of the cost scales treating the weighting factors as parameters. Then by using the search technique described in Chapter (1) we can search for the optimum values of the weighting factors.

For the problem under discussion if we let the cost scales be the state variables of the sectors, the goals of the sectors (objective functions) shall be expressed as follows:

$$\begin{aligned} \phi_{(\ell,m)} = & \sum_{j=1}^{q_\ell} C'_{(\ell,m)(\ell,j)} X_{(\ell,j)} \\ & \quad \vdots \\ + & \sum_{j=1}^{q_K} C'_{(\ell,m)(K,j)} X_{(K,j)} \end{aligned}$$

$$\ell = 1,\ldots,K\,, \qquad m = 1,\ldots,q_\ell \tag{5.27}$$

Let us denote the part of the Hamiltonian $H_{(\ell,m)}$ which depends on the strategy of sector (ℓ,m) by $H_{(\ell,m)(\ell,m)}$, which is the only part subjected to maximization. Using equation (5.10) we get,

$$\begin{aligned} H_{(\ell,m)(\ell,m)} = & \sum_{j=1}^{q_\ell} (AU_{(\ell,m)(\ell,j)} X_{(\ell,m)})' P_{(\ell,m)(\ell,j)} \\ & \quad \vdots \\ + & \sum_{j=1}^{q_K} (AU_{(\ell,m)(K,j)} X_{(\ell,m)})' P_{(\ell,m)(\ell,j)} \end{aligned}$$

$$\ell = 1,\ldots,K\,, \qquad m = 1,\ldots,q_\ell \tag{5.28}$$

After some modifications the expression $H_{(\ell,m)(\ell,m)}$ takes the form,

$$H_{(\ell,m)(\ell,m)} = x_{(\ell,m)_1} \cdot \psi_1(u_{(\ell,m)_1(k,p)_\rho} \cdot P_{(\ell,m)}) \\ \vdots \\ + x_{(\ell,m)_\beta} \cdot \psi_\beta(u_{(\ell,m)_\beta(k,p)_\rho} \cdot P_{(\ell,m)}) \qquad (5.29)$$

where $\psi_{\bar{\gamma}}$'s ($\bar{\gamma} = 1,\ldots,\beta$) are scalar functions. Here β denotes $n_{(\ell,m)}$. We shall use this simpler notation from here on.

From the constraints given in equation (5.26) we get,

$$\max_{U_{(\ell,m)(k,p)_\rho} \varepsilon U_{(\ell,m)}} H_{(\ell,m)(\ell,m)} = \max_{u_{(\ell,m)_1(k,p)_\rho} \varepsilon U_{(\ell,m)}} x_{(\ell,m)_1} \psi_1 \\ \vdots \\ + \max_{u_{(\ell,m)_\beta(k,p)_\rho} \varepsilon U_{(\ell,m)}} x_{(\ell,m)_\beta} \psi_\beta \qquad (5.30)$$

Let us assume that, the conditions,

$$x_{(\ell,m)_1} \geq 0, \ldots\ldots, x_{(\ell,m)_\beta} \geq 0$$

are satisfied in the interval [0,T]. This is a realistic assumption from a practical point of view. For instance if the system in equation (5.25) represents sectors of some economy, then the production of a sector will no doubt be positive. Consequently, for $u^*_{(\ell,m)_{\bar{\gamma}}(k,p)_\rho}$ to be the optimal value of $u_{(\ell,m)_{\bar{\gamma}}(k,p)_\rho}$ the corresponding function $\psi_{\bar{\gamma}}$ must attain their maximum,

$$\psi_{\bar{\gamma}}(u^*_{(\ell,m)_{\bar{\gamma}}(k,p)_\rho} \cdot P_{(\ell,m)}) = \max_{u_{(\ell,m)_{\bar{\gamma}}(k,p)_\rho} \varepsilon U_{(\ell,m)}} \psi_{\bar{\gamma}}(u_{(\ell,m)_{\bar{\gamma}}(k,p)_\rho} \cdot P_{(\ell,m)})$$

$$\bar{\gamma} = 1,\ldots,\beta \qquad (5.31)$$

On the basis of the linearity of $dX_{(\ell,m)}/dt$ with respect to X, the functions $dP_{(\ell,m)}/dt$ do not depend on X. The boundary conditions for $P_{(\ell,m)}$ are,

$$\left.\begin{array}{lll} P_{(\ell,m)(\ell,j)} = C_{(\ell,m)(\ell,j)} & & j = 1,\ldots,q_\ell \\ \vdots & & \vdots \\ P_{(\ell,m)(K,j)} = C_{(\ell,m)(K,j)} & & j = 1,\ldots,q_K \end{array}\right\} \quad (5.32)$$

By integrating the system of differential equations of the adjoint system with the boundary conditions given in (5.32), the optimal strategies for each sector are determined from equation (5.31), $\ell = 1,\ldots,K$ and $m = 1,\ldots,q_\ell$. The optimal strategies are used to determine $x^*_{(\ell,m)}$ ($\ell = 1,\ldots,K$ and $m = 1,\ldots,q_\ell$) by the forward integration of equation (5.25). The search technique used in Chapter (1) may be used to determine the optimum values of the C's in equation (5.27) corresponding to the maximization of any objective functions ϕ's. The problem considered in this chapter will be the subject for future research.

REFERENCES

[1] Case, J.H., "On Some Differential Games in Economics", MRC, U.S. Army, The University of Wisconsin, Technical Summary, Report #874, May 1968.

[2] Karvovskiy, G.S. and Kuznetsov, A.D., "The Maximum Principle in Differential Games for N players", Engineering Cybernetics, No. 6, pp. 10-13, Nov.-Dec. 1966.

CHAPTER (6)

CONCLUSIONS

6.1 Summary

In this thesis, we have been concerned with the optimization of systems with several cost functionals. Both single control and multi-control cases have been treated. While the discussion has been mostly restricted to deterministic systems, some results for systems with sudden changes in parameters are included.

The main contribution of this thesis can be summarized as follows:

It has been shown that the optimization of a system with respect to an objective function (which is expressed as a function of several given cost functionals) is embedded in the linear combination problem (i.e.) an optimization problem in which the performance criterion is a linear combination of the given cost functionals. According to the technique proposed in this thesis, the optimal controls for the linear combination problem are determined as a function of the weighting factors, and the optimum values of these weighting factors are obtained by a search technique for the given objective function.

There are two main advantages of this technique,

1. The solution of the linear combination problem can be used for any number of different objective functions (which are functions of the same given cost functionals). This technique can handle even case where the given objective function attains its extremum on the boundary of π.

2. While previously reported techniques are mostly confined to cases where the single cost functional is expressed as a single integral, the technique proposed in the thesis is not restricted to such cases (see example in Chapter (3)).

Another contribution of this thesis is the development of the modified accelerating step search technique used to solve the example in Chapter (3). Although it is not carried out in this thesis, this technique can be used to solve numerical examples in Chapter (4) also where the system is subjected to jump variation in its parameters.

Perhaps a few words about the computational time required may be in order here. It is true that the computational time required to determine the entire manifold π is high. However, it must be remembered that this will be necessary only when the optimization has to be carried out with respect to many objective functions. In such a case, the time required to determine π is compensated by the fact that the linear combination problem need be solved only once.

In the case where the optimization has to be carried out with respect to one objective function only, it is not necessary to determine the entire manifold π. Thus the computational time will be reduced.

6.2 Suggestions for Future Work

The preceding discussion is not intended to suggest that all unsolved problems have been treated in this thesis. The following are some of the unsolved questions which come to the author's mind. There must be several others.

1. In Chapter (2), it is assumed that the form of the integrands, assuming g_k's are $\equiv 0$, l_k's do not change during the course of the operation of the system. In other words,

$$l_k = l_k^{(1)} \qquad t\varepsilon[t_o,T]$$
$$k = 1,\ldots,N$$

However, situations may arise where this assumption may not be valid. In other words,'

$$l_k = l_k^{(1)} \qquad t\varepsilon[t_o,t_1)$$

and

$$l_k = l_k^{(2)} \qquad t\varepsilon(t_1,T]$$
$$k = 1,\ldots,N$$

where $t_o < t_1 < T$, and is fixed.

The solution of such problems needs to be investigated.

2. In the usal formulation of the Pursuit-Evasion problem it is assumed that the dynamics of the evader is independent of the dynamics of the pursuer. However, it may be useful to consider situations where the pursuer has some effect on the dynamics of the evader and vice versa. Such a problem can be handled by using the technique proposed in Chapter (3). A stochastic version of the preceding problem is also worth considering.

3. The work reported in Chapter (5) of this thesis relating to multi-control optimization problems should be pursued further by applying the technique to specific practical examples.

4. For the multi level system treated in Chapter (5), it may be interesting to explore the feasibility of using Dynamic Programming to solve the optimization problem. The fact that the system not only has multi levels (which may be thought of as stages for purposes of Dynamic Programming) but also many sectors at each level might pose difficulties.

5. The question of using the Functional Analysis approach to the problems treated in this thesis also offers a rich area for future research.

APPENDIX

Computer Programs (IBM 360/67 System)

```
FORTRAN IV G LEVEL 1, MOD 4              MAIN              DATE = 70241              21/50/42

      C     THIS PROGRAM IS CONSIDERED AS A TYPICAL EXAMPLE FOR ONE DIMENSIONAL
      C     SEARCH.THE SYSTEM CONSIDERED IS REPRESENTED BY,
      C           X1'=X2
      C           X2'=U
      C     WITH INITIAL STATE(1,1) AND FREE FINAL TIME.X1(T)=X2(T)=0.
      C
      C     COST SCALES ARE,
      C           OT=TIME ELAPSED TO REACH (0,0),T
      C           OF=FUEL CONSUMED TO REACH (0,0),F
      C
      C           UFO=OBJECTIVE FUNCTION,(OT**R)*OF,R GREATER THAN ONE.
      C
      C     A=R
      C     B=C2/C1
      C     DEL1=INCREMENT OF (B)
      C     DEL2=INCREMENT OF (A)
      C     KK=NO OF BISECTION STEPS=25
      C     KA=NO OF DISCRETE VALUES OF B,LESS THAN 100
      C     KB=NO OF DISCRETE VALUES OF A,LESS THAN 100
      C     THE MAIN PROGRAM MUST BE ASSOCIATED WITH A SUBROUTINE 'PAR'.
      C     SUBROUTINE 'PAR' COMPUTES 'OT'AND'OF'GIVEN C2/C1.
      C     MAIN PROGRAM AND SUBROUTINE ARE DOUBLE PRECISION9
            REAL*8 B,F,T,C,A,UF,UFM,DDX,B1,B2,B3,OF1,OF2,OF3,OT1,OT2,OT3,UFO1,
           XUFO2,UFO3,AA,BB
            DIMENSION F(100),T(100),C(100),UF(100)
            READ(5,100)DEL1,DEL2,A,B,KA,KB
        100 FORMAT(4F16.8,2I3)
            DO 1 J=1,KB
            CALL PAR(B,F(J),T(J))
            C(J)=B
          1 B=B+DEL1
            DO 5 I=1,KA
            DO 2 J=1,KB
            UF(J)=(T(J)**A)*F(J)
          2 WRITE(6,8)C(J),F(J),T(J),UF(J),A
          8 FORMAT(5X,6F14.8)
            UFM=UF(1)
            DO 3 J=1,51
            IF(UF(J)-UFM)4,3,3
          4 UFM=UF(J)
            IC=J
          3 CONTINUE
            B2=(IC-1)*DEL1
            DO 39 KK=1,25
            B1=B2-DEL1/(2.00**KK)
            B3=B2+DEL1/(2.000**KK)
            CALL PAR(B1,OF1,OT1)
            CALL PAR(B2,OF2,OT2)
            CALL PAR(B3,OF3,OT3)
            UFO1=(OT1**A)*OF1
            UFO2=(OT2**A)*OF2
            UFO3=(OT3**A)*OF3
            AA=DABS(UFO2-UFO1)
            BB=DABS(UFO3-UFO2)
            IF((AA+BB)-5.0*(10.0**(-16)))35,35,36
```

```
FORTRAN IV G LEVEL 1, MOD 4            MAIN            DATE = 79241          21/50/42

              36 IF(UFO1-UFO2)30,31,32
              30 IF(UFO2-UFO3)33,33,35
              31 IF(UFO2-UFO3)34,35,35
              32 IF(UFO2-UFO3)34,34,42
              33 B2=B1
                 GO TO 39
              34 B2=B2
                 GO TO 39
              42 B2=B3
              39 CONTINUE
              35 CALL PAR(B2,OF2,OT2)
                 UFO2=(OT2**A)*OF2
                 B1=1.00000000000000/B2
                 B1=B1*(OT2**A)
                 B3=A*(OT2**(A-1.00))*OF2
         C       CHANGE(UFO)/CHANGE(OT)=-B1+B3
                 WRITE(6,10)KK,B1,B3
              10 FORMAT(//,5X,I4,2F14.8)
                 WRITE(6,8)B2,OF2,OT2,UFO2,A
                 WRITE(6,11)
              11 FORMAT(1H1)
               5 A=A+DEL2
                 END
```

```
FORTRAN IV G LEVEL 1, MOD 4             PAR             DATE = 70241            21/50/42

                SUBROUTINE PAR(B,OF,OT)
                REAL*8 B,OF,OT,D,C
                D=0.5+2.0*B
                C=DSQRT(3.0/(1.0+2.0*D))
                OF=1.0+2.0*C
                OT=1.0+(1.5+D)*C
                RETURN
                END
```

```
FORTRAN IV G LEVEL  1, MOD 4                 MAIN               DATE = 70241            21/49/55
      C      THIS PROGRAM IS CONSIDERED AS A TYPICAL EXAMPLE FOR TWO DIMENSIONAL
      C      SEARCH.THE SEARCH TECHNIQUE USED IS THE ACCELERATING STEP TECHNIQUE
      C
      C      THE SYSTEM CONSIDERED IS REPRESNTED BY,
      C              X1'=X2
      C              X2'=U
      C      WITH INITIAL STATE(1,0)AND FREE TERMINAL TIME.X1(T)=X2(T)=0.
      C
      C      COST SCALES ARE,
      C              OT..=TIME ELAPSED TO REACH (0,0)
      C              OF..=FUEL CONSUMED TO REACH (0,0)
      C              OE..=ENERGY DISSIPATED TO REACH (0,0)
      C
      C              OBJECTIVE FUNCTION=10.*(OT..-2.)**2+OF..+OE..
      C                                =COST OR XCOST
      C
      C      OS1=C2/C1
      C      OS2=C3/C1
      C      DDX=INCREMENT IN OF.. DIRECTION
      C      DDY=INCREMENT IN OE.. DIRECTION
      C      XOI=INITIAL STATE
      C
      C      THE PROGRAM IS CONTROLLED TO THREE ITERATIONS,(CAN BE CHANGED).
      C      THE STEP SIZE IN EACH ITERATION IS= 10.**(-(IST(K))),K=1,2,3.
      C      THE PROGRAM IS ASSOCIATED WITH A SUBROUTINE 'PAR'.SUBROUTINE 'PAR'
      C      COMPUTES OS1,OS2,OT..,AND XCOST ASSUMING OF.. AND OE.. ARE GIVEN.
      C      XOI=1.,IF XOI IS CHANGED, 'PAR' MUST BE CHANGED ACCORDINGLT.
             REAL*8 OS1,OS2,XOI,OF,OE,OT,GRAD1,GRAD2,COST,XGRAD1,XGRAD2,XCOST,D
            XDX,DDY,DX,DY,OFN,OEN,OTN,Y,A,B,OFC,OEC,OF1,OF2,OF3,OE1,OE2,OE3,OT1
            X,OT2,OT3,OS11,OS12,OS13,OS21,OS22,OS23,XCOST1,XCOST2,XCOST3
             DIMENSION OF(3),OE(3),OT(3),GRAD1(3),GRAD2(3),COST(3),IS(3),IST(3)
             READ(5,500)OS1,OS2,XOI,OF(1),OE(1),OT(1),GRAD1(1),GRAD2(1),COST(1)
            X,IST(1),IST(2),IST(3)
         500 FORMAT(9F12.8,3I3)
             XGRAD1=-GRAD1(1)
             XGRAD2=-GRAD2(1)
             XCOST=COST(1)
             WRITE(6,101)
         101 FORMAT(10X,'GRAD1',9X,'GRAD2',10X,'FUEL',9X,'ENERGY',9X,'TIME',9X,
            X'LENGTH',6X,'-EPS2/EPS1',4X,'-EPS3/EPS1',/)
             DO 15 JJ=1,3
      C
      C      EQUAL STEP SEARCH
      C
             WRITE(6,300)JJ
         300 FORMAT(5X,'ITERATION STEP NUMBER (',I1,')')
             JC=0
             WRITE(6,8)GRAD1(1),GRAD2(1),OF(1),OE(1),OT(1),XCOST,OS1,OS2
           8 FORMAT(5X,8F14.8)
             K=1
           3 IF(XGRAD1)10,11,12
          10 DDX=-10.0**(-(IST(JJ)))
             DDY=DDX*(XGRAD2/XGRAD1)
             GO TO 13
          11 IF(XGRAD2)14,18,16
```

```
 14   DDX=0.000000000
      DDY=-10.0**(-(IST(JJ)))
      GO TO 13
 16   DDX=0.000000000000
      DDY=10.0**(-(IST(JJ)))
      GO TO 13
 12   DDX=10.0**(-(IST(JJ)))
      DDY=DDX*(XGRAD2/XGRAD1)
 13   DO 1 I=1,1000
      DX=DDX*I
      DY=DDY*I
      OFN=OF(K)+DX
      OEN=OE(K)+DY
      Y=XCOST
      CALL PAR(OFN,OEN,OS1,OS2,OTN,XCOST)
      OS1=-OS1
      OS2=-OS2
      IF(XCOST-Y)1,2,2
  1   CONTINUE
C
C     BISECTION SEARCH
C
  2   OFC=OFN-DDX
      OEC=OEN-DDY
      DO 39 KK=1,12
      OF2=OFC
      OF1=OF2-DDX/(2.00000000000**KK)
      OF3=OF2+DDX/(2.0000000000000**KK)
      OE2=OEC
      OE1=OE2-DDY/(2.000000000000000**KK)
      OE3=OE2+DDY/(2.0000000000000000**KK)
      CALL PAR(OF1,OE1,OS11,OS21,OT1,XCOST1)
      CALL PAR(OF2,OE2,OS12,OS22,OT2,XCOST2)
      CALL PAR(OF3,OE3,OS13,OS23,OT3,XCOST3)
      A=DABS(XCOST2-XCOST1)
      B=DABS(XCOST3-XCOST2)
      IF((A+B)-5.0*(10.0**(-16)))35,35,36
 36   IF(XCOST1-XCOST2)30,31,32
 30   IF(XCOST2-XCOST3)33,40,40
 31   IF(XCOST2-XCOST3)34,40,40
 32   IF(XCOST2-XCOST3)34,34,42
 33   OFC=OF1
      OEC=OE1
      GO TO 39
 34   OFC=OF2
      OEC=OE2
      GO TO 39
 42   OFC=OF3
      OEC=OE3
      GO TO 39
 40   WRITE(6,41)
 41   FORMAT(2X,'SOMETHING WRONG IN THE BISECTION SEARCH')
      GO TO 35
 39   CONTINUE
 35   OFN=OFC
```

```
FORTRAN IV G LEVEL 1, MOD 4              MAIN              DATE = 70241          21/49/55

          OEN=OEC
          CALL PAR(OFN,OEN,OS1,OS2,OTN,XCOST)
          OS1=-OS1
          OS2=-OS2
          IF(JC-1)22,23,23
      22  IS(K)=I
          IF(K-2)4,5,5
       4  OF(2)=OFN
          OE(2)=OEN
          OT(2)=OTN
          GRAD1(2)=20.0000000000*(OTN-2.00000000000)*OS1+1.0000000000000
          GRAD2(2)=20.0000000000*(OTN-2.000000000)*OS2+1.000000000000
          XGRAD1=-GRAD1(2)
          XGRAD2=-GRAD2(2)
          WRITE(6,8)GRAD1(2),GRAD2(2),OF(2),OE(2),OT(2),XCOST,OS1,OS2
          K=2
          GO TO 3
       5  WRITE(6,102)OFN,OEN,OTN,XCOST,OS1,OS2
     102  FORMAT(33X,6F14.8)
          XGRAD1=OFN-OF(1)
          XGRAD2=OEN-OE(1)
          XCOST=COST(1)
          JC=1
          K=JC
          GO TO 3
      23  GRAD1(1)=20.000000000*(OTN-2.000000000)*OS1+1.0000000000000000
          GRAD2(1)=20.0000000000*(OTN-2.0000000000)*OS2+1.000000000000
          IS(3)=I
          WRITE(6,8)GRAD1(1),GRAD2(1),OFN,OEN,OTN,XCOST,OS1,OS2
          WRITE(6,104)IS(1),IS(2),IS(3)
     104  FORMAT(5X,I5,5X,I5,5X,I5)
          A=DABS(GRAD1(1))+DABS(GRAD2(1))
          IF(A-0.0000000000005)17,17,24
      24  OF(1)=OFN
          OE(1)=OEN
          OT(1)=OTN
          COST(1)=XCOST
          XGRAD1=-GRAD1(1)
          XGRAD2=-GRAD2(1)
      15  WRITE(6,100)
     100  FORMAT(///)
          GO TO 17
      18  A=-XGRAD1
          B=-XGRAD2
          WRITE(6,9)A,B,OFN,OEN,OTN,XCOST,OS1,OS2,K,JC
       9  FORMAT(5X,8F14.8,2I2)
          GO TO 200
      17  A=-XGRAD1
          B=-XGRAD2
          WRITE(6,400)A,B
     400  FORMAT(1H1,2F20.14)
     200  A=A
          END
```

FORTRAN IV G LEVEL 1, MOD 4 PAR DATE = 70241 21/49/55

```
      SUBROUTINE PAR(OFN,OEN,OS1,OS2,OTN,XCOST)
      REAL*8 OFN,OEN,OTN,OS1,OS2,XCOST
      OS2=3.0*(1.0-OEN/OFN)
      OS1=2.0/(OFN*OFN)+OS2*OS2/6.0-OS2-0.500
      OTN=(1.0+OS1+OS2)*OFN
      XCOST=10.000000000000*(OTN-2.00000000)*(OTN-2.00000000)+OFN+OEN
      RETURN
      END
```

```
FORTRAN IV G LEVEL 1, MOD 4              MAIN              DATE = 70241          22/08/09

      C      THIS PROGRAM IS CONSIDERED AS A TYPICAL EXAMPLE FOR A SADDLE-POINT
      C      SEARCH.THE TECHNIQUE USED IS A MODIFICATION OF THE ACCELERATING
      C      STEP SEARCH TECHNIQUE,REFER TO THESIS.
      C      THE SYSTEM CONSIDERED IS A FIRST ORDER SYSTEM REPRESENTED BY,
      C              X'=A*X+B*U+C*V
      C
      C      THE COST SCALES COSIDERED ARE,
      C              EX1=0.5*(FINAL STATE)**2
      C              EX2=0.5*(ENERGY DISSIPATED BY CONTROL U)
      C              EX3=0.5*(ENERGY DISSIPATED BY CONTRCL V)
      C
      C      OBJECTIVE=MIN MAX(EX1+(EX3)/(EX2+0.5))   ,V MIN AND U MAX
      C               =MIN MAX(OF),(SUBROUTINE'PAR')
      C
      C      EPS1=C2/C1
      C      EPS2=C3/C1
      C      KK=+1OR-1       +1 FOR STEPS IN GRADIENT DIRECTION
      C                      -1 FOR STEPS OPPOSITE TO GRADIENT DIRECTION
      C      GRAD1=D(OF)/D(EX2)
      C      GRAD2=D(OF)/D(EX3)
      C      OF1=(GRAD1*GRAD1+GRAD2*GRAD2),(SUBROUTINE'PAR')
      C      THE PROGRAM IS CONTROLLED TO THREE ITERATIONS(CAN BE CHANGED).
      C      THE STEP SIZE IN EACH ITERATION=(10.**(-IC(K))),K=1,2,3
      C
      C      THE MAIN PROGRAM IS ASSOCIATED WITH THREE SUBROUTINES,'STEP','PAR'
      C      ,AND'BISEC'.SUBROUTINE'STEP' IS TO DETERMINE THE STEPS IN THE EX2
      C      AND EX3 DIRECTIONS.SUBROUTINE 'PAR' IS TO DETERMINE EPS1,EPS2,OF,
      C      OF1 ASSUMING EX2 AND EX3 ARE GIVEN.SUBROUTINE 'BISEC' IS A
      C      BISECTION SEARCH.
             REAL*8 XS1,XS2,XS3,EPS1,EPS2,OF,X1,X2,X3,GRAD1,GRAD2,DX2,DX3,D2,D3
            X,X1N,X2N,X3N,Y,OFT,YY,OFF
             REAL*8 A,B,C,T,XO,XS,CS,BS,Q1,Q2,Q4,Q5,Q6,Q7,Q8,Q9,Q10,OF1,A1,A2
             REAL*8 DEXP,DABS,DSQRT
             DIMENSION X1(3),X2(3),X3(3),IC(3),ICC(3)
             READ(5,500)A,B,C,XO,T,EPS1,EPS2,IC(1),IC(2),IC(3)
         500 FORMAT(7F14.8,3I3)
             XS=XO*XO
             CS=C*C
             BS=B*B
             Q1=DEXP(-A*T)
             Q2=DEXP(-2.*A*T)
             Q4=1.00000
             Q5=EPS2*CS-EPS1*BS
             Q6=Q5*Q4
             Q7=Q6+(1.00000-Q6)*Q2
             Q8=1.000000/Q7
             Q9=Q8*Q8
             Q10=XS*Q9*Q4*(1.00000000-Q2)
             XS2=0.5000000*BS*EPS1*EPS1*Q10
             XS3=0.5000000*CS*EPS2*EPS2*Q10
             DO 15 IK=1,3
             CALL PAR (XS2,XS3,EPS1,EPS2,XS1,OF,OF1)
             X2(1)=XS2
             X3(1)=XS3
             X1(1)=XS1
```

```
FORTRAN IV G LEVEL 1, MOD 4              MAIN               DATE = 70241            22/08/09

      A1=X3(1)
      A2=X2(1)+0.5000000000000000
      GRAD1=1.0000000000/EPS1-A1/(A2*A2)
      GRAD2=-1.0000000000/EPS2+1.0000000000000/A2
      WRITE(6,8)GRAD1,GRAD2,X1(1),X2(1),X3(1),OF,EPS1,EPS2
    8 FORMAT(5X,8F14.8)
      KK=1
      CALL STEP(GRAD1,GRAD2,KK,IC(IK),DX2,DX3)
      OFT=OF1
      YY=OFT
      XS2=XS2+DX2
      XS3=XS3+DX3
      CALL PAR (XS2,XS3,EPS1,EPS2,XS1,OFF,OFT)
      IF(OFT-YY)11,12,12
   12 KK=-1
      DX2=-DX2
      DX3=-DX3
   11 DO 1 I=1,1000
      D2=DX2*I
      D3=DX3*I
      X2N=X2(1)+D2
      X3N=X3(1)+D3
      Y=OF1
      CALL PAR (X2N,X3N,EPS1,EPS2,X1N,OF,OF1)
      IF(OF1-Y)1,2,2
    1 CONTINUE
    2 CALL BISEC (X2N,X3N,DX2,DX3)
  100 ICC(1)=I
      X2(2)=X2N
      X3(2)=X3N
      CALL PAR (X2(2),X3(2),EPS1,EPS2,X1(2),OF,OF1)
      A1=X3(2)
      A2=X2(2)+0.5000000000000
      GRAD1=1.000000000/EPS1-A1/(A2*A2)
      GRAD2=-1.000000000/EPS2+1.000000000/A2
      WRITE(6,8)GRAD1,GRAD2,X1(2),X2(2),X3(2),OF,EPS1,EPS2
      KK=1
      CALL STEP (GRAD1,GRAD2,KK,IC(IK),DX2,DX3)
      OFT=OF1
      YY=OFT
      A1=X3(2)+DX3
      A2=X2(2)+DX2
      CALL PAR (A2,A1,EPS1,EPS2,X1N,OFF,OFT)
      IF(OFT-YY)110,120,120
  120 KK=-1
      DX2=-DX2
      DX3=-DX3
  110 DO 3 I=1,1000
      D2=DX2*I
      D3=DX3*I
      X2N=X2(2)+D2
      X3N=X3(2)+D3
      Y=OF1
      CALL PAR (X2N,X3N,EPS1,EPS2,X1N,OF,OF1)
      IF(OF1-Y)3,5,5
```

```
FORTRAN IV G LEVEL 1, MOD 4                MAIN              DATE = 70241             22/08/09

    3 CONTINUE
    5 CALL BISEC (X2N,X3N,DX2,DX3)
  200 ICC(2)=I
      X2(3)=X2N
      X3(3)=X3N
      CALL PAR (X2(3),X3(3),EPS1,EPS2,X1(3),OF,OF1)
      GRAD1=X2(3)-X2(1)
      GRAD2=X3(3)-X3(1)
      WRITE(6,9)X1(3),X2(3),X3(3),OF,EPS1,EPS2
    9 FORMAT(33X,6F14.8)
      KK=1
      CALL STEP(GRAD1,GRAD2,KK,IC(IK),DX2,DX3)
      CALL PAR(X2(1),X3(1),EPS1,EPS2,X1(1),OF,OF1)
      DO 4 I=1,1000
      D2=DX2*I
      D3=DX3*I
      X2N=X2(1)+D2
      X3N=X3(1)+D3
      Y=OF1
      CALL PAR (X2N,X3N,EPS1,EPS2,X1N,OF,OF1)
      IF(OF1-Y)4,6,6
    4 CONTINUE
    6 CALL BISEC (X2N,X3N,DX2,DX3)
      ICC(3)=I
      CALL PAR (X2N,X3N,EPS1,EPS2,X1N,OF,OF1)
      A1=X3N
      A2=X2N+0.500000000000
      GRAD1=1.000000000000/EPS1-A1/(A2*A2)
      GRAD2=-1.000000000000/EPS2+1.00000000000/A2
      WRITE(6,8)GRAD1,GRAD2,X1N,X2N,X3N,OF,EPS1,EPS2
      WRITE(6,7)ICC(1),ICC(2),ICC(3)
    7 FORMAT(2X,3I10)
      XS2=X2N
   15 XS3=X3N
      END
```

```
FORTRAN IV G LEVEL 1, MOD 4          STEP          DATE = 70241          22/08/09

         SUBROUTINE STEP (A,B,KK,I,DA,DB)
         REAL*8 A,B,DA,DB,C,D,E
         REAL*8 DEXP,DABS,DSQRT
         IF(A)10,11,10
      11 IF(DABS(A)-0.0000000000000005)20,20,21
      21 IF(B)10,18,10
      10 C=DSQRT(1.+(B/A)*(B/A))
         E=DSQRT(1.+(A/B)*(A/B))
         D=10.0**(-I)
         DA=KK*D*(A/DABS(A))/C
         DB=KK*D*(B/DABS(B))/E
         GO TO 19
      20 DA=0.000000000000000
         DB=KK*D
         GO TO 19
      18 WRITE(6,8)
       8 FORMAT(2X,'THE GRADIENT IS VANISHED')
      19 RETURN
         END
```

```
FORTRAN IV G LEVEL 1, MOD 4            PAR            DATE = 70241            22/08/09

      SUBROUTINE PAR(EX2,EX3,EPS1,EPS2,EX1,OF,OF1)
      REAL*8 A,B,C,T,XO,XS,CS,BS,Q1,Q2,Q3,Q4,EPS1,EPS2,Q5,Q6,Q7,Q8,Q9,EX
     X1,Q10,EX2,EX3,OF,R1,R2,R3,R4,R5,GRAD1,GRAD2,OF1,A1,A2
      REAL*8 DEXP,DABS,DSQRT
      A=0.50000000
      B=0.050000000000000
      C=0.051000000000000
      T=2.00000000
      XO=1.500000000
      XS=XO*XO
      CS=C*C
      BS=B*B
      Q1=DEXP(-A*T)
      Q2=DEXP(-2.00*A*T)
      Q4=1.00000/(2.000000*A)
      R1=(B/C)*DSQRT(EX3/EX2)
      R2=BS*XS*0.500000000*(1.0000000-Q2)
      R3=(R1*CS-BS)*(1.000000000000-Q2)
      R4=DSQRT(EX2/R2)
      R5=1.0000000/R4
      EPS1=Q2*(1.00000000/(R5-R3))
      EPS2=EPS1*R1
      Q5=EPS2*CS-EPS1*BS
      Q6=Q5*Q4
      Q7=Q6+(1.00000-Q6)*Q2
      Q8=1.000000/Q7
      Q9=Q8*Q8
      EX1=0.50000*XS*Q2*Q9
      A1=EX3
      A2=EX2+0.5000000000000
      GRAD1=1.0000000000/EPS1-A1/(A2*A2)
      GRAD2=-1.0000000000/EPS2+1.0000000000/A2
      OF1=GRAD1*GRAD1+GRAD2*GRAD2
      OF=EX1+A1/A2
      RETURN
      END
```

```
      SUBROUTINE BISEC (A,B,DA,DB)
      REAL*8 A,B,DA,DB,DX,DY,XC,YC,X1,X2,X3,Y1,Y2,Y3,OS11,OS21,OS22,OS13
     X,OS23,Z1,Z2,Z3,V1,V2,V3,OS12,VO1,VO2,VO3
      REAL*8 DEXP,DABS,DSQRT
      DX=DA
      DY=DB
      XC=A
      YC=B
      DO 39 K=1,40
      X2=XC
      X1=X2-DX/(2.000000000**K)
      X3=X2+DX/(2.00000000000**K)
      Y2=YC
      Y1=Y2-DY/(2.000000000**K)
      Y3=Y2+DY/(2.000000000**K)
      CALL PAR (X1,Y1,OS11,OS21,Z1,VO1,V1)
      CALL PAR (X2,Y2,OS12,OS22,Z2,VO2,V2)
      CALL PAR (X3,Y3,OS13,OS23,Z3,VO3,V3)
      A=DABS(V2-V1)
      B=DABS(V3-V2)
      IF((A+B)-5.0*(10.0**(-16)))40,40,36
   36 IF(V1-V2)30,31,32
   30 IF(V2-V3)33,33,40
   31 IF(V2-V3)34,40,42
   32 IF(V2-V3)34,34,42
   33 XC=X1
      YC=Y1
      GO TO 39
   34 XC=X2
      YC=Y2
      GO TO 39
   42 XC=X3
      YC=Y3
   39 CONTINUE
   40 A=XC
      B=YC
      WRITE(6,8)K
    8 FORMAT(2X,I10)
      RETURN
      END
```

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